INTEGRATION THEOREMS FOR GAGES AND DUALITY FOR UNIMODULAR GROUPS(1)

BY

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INTRODUCTION

Let $\mathfrak{A}$ be a W^*-algebra on a Hilbert space $\mathfrak{H}$. In [10], Segal presents a kind of integration theory in which, instead of integrating functions on a space with respect to a measure, one integrates possibly unbounded operators affiliated with $\mathfrak{A}$ with respect to a "gage" on $\mathfrak{A}$. In the present work, we investigate analogues of certain portions of ordinary integration theory which Segal did not have occasion to develop in [10], and then we apply the theory of gage spaces to harmonic analysis and duality of unimodular groups.

One of the most important ways of constructing an integral in ordinary measure theory is by starting from a positive linear functional on some algebra of bounded functions. In §1, the corresponding construction of gages is considered. Theorem 1 gives conditions under which a positive central linear functional on a self-adjoint algebra of bounded operators yields a gage on the W^*-algebra it generates. This theorem is applied in §7 to the construction of product gages and in §9 to the construction of the dual gage for a unimodular group. The question of what can be said about the extension of noncentral positive linear functionals naturally arises. A special case is dealt with in §2.

In [10], convergence nearly everywhere (n.e.) is defined for gage spaces. In §3, a notion of convergence in measure is introduced. It is found to have the usual relations to convergence n.e. and mean convergence. In §4, several dominated convergence theorems are proved together with a version of Fatou's lemma. Convergence in measure is most often used in probability spaces. In later work than [10], Segal has used a notion of convergence in probability for self-adjoint operators on a probability gage space, which is defined differently from our convergence in measure. However, the two are shown to be equivalent in §5. In §6, a monotone convergence theorem is proved in which the convergence takes place in a more "pointwise" sense than in Segal's monotone convergence theorem [10, Corollary 13.2]. This is useful in connection with the Fubini theorem in §7.

The Fubini theorem in §7 differs from Segal's Fubini theorem [10, Corollary 22.1] in that the marginal integral is explicitly constructed. This is useful in proving Lemma 9.12 about convolutions in the dual gage space to a

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unimodular group. §8 defines the tensor product of two closed unbounded operators in a natural way. It closes with Theorem 8.4 which asserts that the marginal integral of the product of a bounded operator with the tensor product of two integrable operators is what it ought to be. It is this sort of marginal integral which occurs in the proof of Lemma 9.12, cited above.

In §9, a global form of the Fourier transform for unimodular groups is considered. The role of measurable functions on the character group in the abelian case is played in the general case by operators measurable with respect to the dual gage space. A number of analogues of standard theorems in harmonic analysis of abelian groups are developed, among them being L_1 inversion theorems. Also, a convolution operation is defined in L_1 of the dual gage space which maps into pointwise multiplication of functions on the group under the inverse transformation. Under this convolution and a companion involution, L_1 of the dual gage space becomes a commutative Banach algebra with an involution. In §10, a duality theorem parallel to the Pontrjagin and Tannaka duality theorems is proved. It asserts that every nonzero $_*$-homomorphism of this Banach algebra with involution into the complex numbers arises from an element of the group.

Throughout the present work, the notations, terminology, and results of [10] are fundamental. Whenever the sum or product of two unbounded measurable operators occurs, the strong sum or strong product [10, Definition 2.2] is understood. For any measurable operator T, we write Re T for $(T+T^*)/2$ and Im T for $(T-T^*)/2$. Unions and intersections of projections are to be interpreted as formed in the lattice of all projections on $\mathfrak{H}$. If M is a linear submanifold of $\mathfrak{H}$, $\overline{M}$ will denote the closure of M. Any closed (possibly unbounded) operator T on $\mathfrak{H}$ may be written uniquely as $T=U|T|$ where $|T|$ is a non-negative self-adjoint operator and U is a partial isometry with initial subspace $[|T|(\mathfrak{H})]^-$ and final subspace $[T(\mathfrak{H})]^-$. In fact $|T| =(T^*T)^{1/2}$. We shall refer to $T=U|T|$ as the polar decomposition of T. Since $T^*=|T|U^*$, we see that $[|T|(\mathfrak{H})]^-=[T^*(\mathfrak{H})]^-$. Thus, it is clear that if T is a closed operator affiliated with $\mathfrak{A}$, then $[T(\mathfrak{H})]^-$ and $[T^*(\mathfrak{H})]^-$ have the same relative dimensionality [10, p. 403] with respect to $\mathfrak{A}$.

I. Extension of positive linear functionals

1. **Central positive linear functionals.** In ordinary integration theory, the Hahn-Kolmogoroff extension theorem [4, Theorem 13. A] gives conditions under which a positive linear functional defined on the algebra of functions which are finite linear combinations of characteristic functions of sets in a Boolean ring is really the integral with respect to a (countably-additive) measure. Similarly, the Riesz-Markoff theorem [4, Theorem 56. D] says that a positive linear functional on the algebra of all continuous functions with a compact support on a locally compact Hausdorff space is really the integral with respect to a regular measure. Theorem 1.1 is an analogous theo-

rem for the noncommutative case; it gives conditions under which a positive linear functional l on a weakly dense subalgebra of a W^*-algebra is the integral with respect to a gage. For this to be true, it must, of course, be assumed that l is central; that is to say, that $l(AB)=l(BA)$ for all A and B in the subalgebra.

THEOREM 1.1. *Let $\mathfrak{A}$ be a self-adjoint operator algebra on a Hilbert space $\mathfrak{K}$ which is generated* algebraically *by its non-negative elements. Let l be a central linear functional on $\mathfrak{A}$ which is non-negative in the sense that $l(A^*A)\geqq 0$ for all* A *in $\mathfrak{A}$. Suppose that:*

(1) *For all A and B in $\mathfrak{A}$, the mapping $T\to l(ATB)$ restricted to the unit ball of $\mathfrak{A}$ is weakly continuous;*

(2) *There exists a directed system $\{E_\nu\}$ of elements of $\mathfrak{A}$ such that for all A in $\mathfrak{A}$, $l(AE_\nu)\to l(A)$ and $l(E_\nu^*AE_\nu)\to l(A)$. Then there exists a gage m on the weak closure $\overline{\mathfrak{A}}$ of $\mathfrak{A}$ such that $\mathfrak{A}\subset L_1(\mathfrak{K}, \overline{\mathfrak{A}}, m)$ and $l(A)=m(A)$ for all A in $\mathfrak{A}$.*

Proof. First we show that if A is any non-negative element of $\mathfrak{A}$, then $l(A)\geqq 0$. For, let $\{p_n\}$ be a sequence of real polynomials vanishing at 0 such that $p_n(\lambda)\to\lambda^{1/2}$ uniformly for $0\leqq\lambda\leqq\|A\|$. Then for fixed ν, $E_\nu^*(p_n(A))^2E_\nu$ remains bounded and $\to E_\nu^*AE_\nu$ weakly as $n\to\infty$. Therefore by hypothesis (1), $l(E_\nu^*AE_\nu)\geqq 0$ for all ν, and hence $l(A)\geqq 0$ by hypothesis (2).

Let $\mathfrak{N}=\{A\in\mathfrak{A}: l(A^*A)=0\}$. Then $\mathfrak{N}$ is a linear manifold by the Schwarz inequality. The inequality:

$$(1) \qquad 0\leqq l(A^*B^*BA)\leqq\|B\|^2 l(A^*A)$$

shows that $\mathfrak{N}$ is a left ideal. Since $l(A^*A)=l(AA^*)$, $\mathfrak{N}$ is invariant under $A\to A^*$. Thus $\mathfrak{N}$ is a two-sided $*$-ideal in $\mathfrak{A}$. It follows that $\mathfrak{D}=\mathfrak{A}/\mathfrak{N}$ is both a $*$-algebra and a pre-Hilbert space, where the inner product of two cosets $A+\mathfrak{N}$ and $B+\mathfrak{N}$ is $l(B^*A)$. Let $\mathfrak{H}$ be the completion of $\mathfrak{K}$ and let J be the conjugation of $\mathfrak{H}$ which extends the involution of $\mathfrak{D}$.

We next show that ($\mathfrak{H}$, J, $\mathfrak{D}$) is a Hilbert algebra [10, Definition 5.1]. That $(ab, c)=(b, a^*c)=(a, cb^*)$ results from the very definition of J and the inner product. The continuity of multiplication is a consequence of inequality (1). The third condition, a nondegeneracy condition, may be replaced by: $\mathfrak{D}^2$ is dense in $\mathfrak{D}$. This holds since $(A+\mathfrak{N})(E_\nu+\mathfrak{N})\to A+\mathfrak{N}$; for

$$l((E_\nu A- A)^*(AE_\nu - A)) \\ = l(E_\nu^*A^*AE_\nu) - l(A^*AE_\nu) - l(E_\nu^*A^*A) + l(A^*A)\to 0.$$

For a in $\mathfrak{D}$, let L_a be the bounded operator on $\mathfrak{H}$ which extends left multiplication by a acting on $\mathfrak{D}$. For A in $\mathfrak{A}$, let $\phi(A)=L_{A+\mathfrak{N}}$. Then ϕ is a norm-decreasing (by inequality (1)) $*$-homomorphism of $\mathfrak{A}$ into the left ring $\mathfrak{L}$ of the Hilbert algebra [10, Definition 5.5]. Let $\mathfrak{A}_1$ be the unit ball of $\mathfrak{A}$ and $\mathfrak{L}_1$, the unit ball of $\mathfrak{L}$. The restriction $\phi|\mathfrak{A}_1$ of ϕ to $\mathfrak{A}_1$ is a weakly continuous map of $\mathfrak{A}_1$ into $\mathfrak{L}_1$. For, the linear functional $T\to((\phi|\mathfrak{A}_1)(T)x, y)$

with x and y in $\mathcal{H}$ is a uniform limit of such linear functionals with x and y in $\mathfrak{D}$; the latter are maps $T \rightarrow l(B^*TA)$ with A and B in $\mathfrak{A}$ and T restricted to $\mathfrak{A}_1$, and they are weakly continuous by hypothesis (1).

Therefore $\phi | \mathfrak{A}_1$ is uniformly continuous with respect to the weak uniform structures on $\mathfrak{A}_1$ and $\mathfrak{L}_1$. By [6], $\mathfrak{A}_1$ is weakly dense in the unit ball $(\overline{\mathfrak{A}})_1$ of $\overline{\mathfrak{A}}$. Since $\mathfrak{L}_1$, being weakly compact, is complete in the weak uniform structure, $\phi | \mathfrak{A}_1$ has a unique extension to a weakly continuous map of $(\overline{\mathfrak{A}})_1$ into $\mathfrak{L}_1$. This extension of $\phi | \mathfrak{A}_1$ can in turn be extended to a unique linear map of $\overline{\mathfrak{A}}$ into $\mathfrak{L}$. We shall again use the letter ϕ for this last map. The new ϕ is a norm-decreasing $*$-homomorphism of $\overline{\mathfrak{A}}$ into $\mathfrak{L}$ which is weakly continuous on $(\overline{\mathfrak{A}})_1$.

Let μ be the canonical gage on $\mathfrak{L}$ [10, Definition 5.7]. If $A \in \mathfrak{A}^2$, $\phi(A) \in L_1(\mathcal{H}, \mathfrak{L}, \mu)$ and $l(A) = \mu(\phi(A))$. For when $A = BC$ with B and C in $\mathfrak{A}$, then $\mu(\phi(BC)) = \mu(\phi(B)\phi(C)) = \mu(L_b L_c) = (c, b^*)$ [10, Corollary 19.1] $= l(BC)$ where we have set $b = B + \mathfrak{N}$ and $c = C + \mathfrak{N}$.

The function $\mu \circ \phi$ is completely additive, non-negative, and unitarily invariant on projections in $\overline{\mathfrak{A}}$. The only further property required of $\mu \circ \phi$ to be a gage is that every projection in $\overline{\mathfrak{A}}$ be the least upper bound of projections on which $\mu \circ \phi$ is finite. To obtain a gage, we define a new function which obviously has this property. If P is a projection in $\overline{\mathfrak{A}}$, let $m(P)$ be the least upper bound of $\mu(\phi(Q))$ as Q runs over all projections $\leqq P$ in $\overline{\mathfrak{A}}$ for which $\mu(\phi(Q))$ is finite. Then m is a gage on $\overline{\mathfrak{A}}$ which agrees with $\mu \circ \phi$ on projections on which $\mu \circ \phi$ is finite. We show that if $A \in \overline{\mathfrak{A}}$ and $\phi(A) \in L_1(\mathcal{H}, \mathfrak{L}, \mu)$, then $A \in L_1(\mathcal{K}, \overline{\mathfrak{A}}, m)$ and $m(A) = \mu(\phi(A))$. For we may assume without loss of generality that A is self-adjoint with spectral resolution $\int_{-\infty}^{+\infty} \lambda dE_\lambda$. The spectral resolution of $\phi(A)$ is then $\int_{-\infty}^{+\infty} \lambda d\phi(E_\lambda)$. By [10, Corollary 12.6], since $\phi(A) \in L_1(\mathcal{H}, \mathfrak{L}, \mu)$,

$$\int_{-\infty}^{+\infty} |\lambda| \, d\mu(\phi(E_\lambda)) < +\infty.$$

It follows from the aforementioned equality between m and $\mu \circ \phi$ on projections on which $\mu \circ \phi$ is finite that

$$\int_{-\infty}^{+\infty} |\lambda| \, dm(E_\lambda) = \int_{-\infty}^{+\infty} |\lambda| \, d\mu(\phi(E_\lambda)) < +\infty.$$

Therefore, again by [10, Corollary 12.6], $A \in L_1(\mathcal{K}, \overline{\mathfrak{A}}, m)$. Now we see that

$$m(A) = \int_{-\infty}^{+\infty} \lambda dm(E_\lambda) = \int_{-\infty}^{+\infty} \lambda d\mu(\phi(E_\lambda)) = \mu(\phi(A)).$$

Putting the last two paragraphs together, we conclude that $\mathfrak{A}^2 \subset L_1(\mathcal{K}, \overline{\mathfrak{A}}, m)$ and $l(A) = m(A)$ for $A \in \mathfrak{A}^2$. Since $\mathfrak{A}$ is generated by its non-negative elements, it is sufficient to show that non-negative elements of $\mathfrak{A}$ are integrable and that l agrees with m on them. To show this, suppose that

A is a non-negative element of $\mathfrak{A}$, and let $p_n(\lambda)$ be a real polynomial vanishing at $\lambda=0$ which approximates $1-e^{-n\lambda}$ within $e^{-2n\|A\|}$ on $0\leqq\lambda\leqq\|A\|$. Then on $0\leqq\lambda\leqq\|A\|$, $\lambda[p_n(\lambda)]^2\to 0$ uniformly, while maintaining the inequality:

$$0 \leqq \lambda[p_n(\lambda)]^2 \leqq \lambda.$$

But

$$l(A[p_n(A)]^2) = m(A[p_n(A)]^2) = \int_0^{\|A\|} \lambda[p_n(\lambda)]^2 dm(E_\lambda)$$

where $A=\int_0^{\|A\|}\lambda dE_\lambda$ is the spectral resolution of A. We will show in the next paragraph that $l(A[p_n(A)]^2)\to l(A)$. If this is granted, Fatou's lemma [4, Theorem 27.F] implies that λ is integrable over $0\leqq\lambda\leqq\|A\|$ with respect to $dm(E_\lambda)$. Then the Lebesgue Dominated Convergence Theorem [4, Theorem 26.D] implies that

$$\int_0^{\|A\|} \lambda[p_n(\lambda)]^2 dm(E_\lambda) \to \int_0^{\|A\|} \lambda dm(E_\lambda) = m(A).$$

We have shown that $A\in L_1(\mathcal{H}, \overline{\mathfrak{A}}, m)$ and that $l(A)=m(A)$.

It remains to prove that l is weakly continuous on $\{T\in\mathfrak{A}: 0\leqq T\leqq A\}$. But

$$\begin{aligned}
|l(E_\nu^*TE_\nu) - l(T)| &\leqq |l((E_\nu^* - I)TE_\nu)| + |l(T(E_\nu - I))| \\
&= |l((E_\nu^* - I)T^{1/2}\cdot T^{1/2}E_\nu)| + |l(T^{1/2}\cdot T^{1/2}(E_\nu - I))| \\
&\leqq [l((E_\nu - I)^*T(E_\nu - I))l(E_\nu^*TE_\nu)]^{1/2} \\
&\quad + [l(T)l((E_\nu - I)^*T(E_\nu - I))]^{1/2} \\
&\qquad \text{by the Schwarz inequality} \\
&\leqq [l((E_\nu - I)^*A(E_\nu - I))l(E_\nu^*AE)]^{1/2} \\
&\quad + [l(A)l((E_\nu - I)^*A(E_\nu - I))]^{1/2} \\
&\qquad \text{by the positivity property of } l.
\end{aligned}$$

Hypothesis (2) implies that this last expression converges to 0. Hence the map $T\to l(T)$ is the uniform limit of maps $T\to l(E_\nu^*TE_\nu)$ on $0\leqq T\leqq A$. The latter maps are weakly continuous by hypothesis (1). Therefore l is weakly continuous on $\{T\in\mathfrak{A}: 0\leqq T\leqq A\}$.

The question of the uniqueness of m in the preceding theorem is settled in Corollary 1.3. It must be noted that the strong closure of a self-adjoint operator algebra is the same as its weak closure [13, part II].

LEMMA 1.2. *Let $(\mathcal{H}, \overline{\mathfrak{A}}, m)$ be a gage space. If $\mathfrak{A}$ is a strongly dense *-subalgebra of $\overline{\mathfrak{A}}$ which is contained in $L_p(\mathcal{H}, \overline{\mathfrak{A}}, m)$ where $p=1$ or 2, then $\mathfrak{A}$ is dense in $L_p(\mathcal{H}, \overline{\mathfrak{A}}, m)$.*

Proof (Adapted from the proof of Theorem 22 in [10]). If $\mathfrak{A}$ were not dense in $L_p(\mathfrak{H}, \overline{\mathfrak{A}}, m)$, then there would exist $T \in L_{p'}(\mathfrak{H}, \overline{\mathfrak{A}}, m)$, where $p'=2$ or ∞ respectively, such that $m(AT)=0$ for all $A \in \mathfrak{A}$ by [10, Corollary 18.1], or [10, Theorem 13] respectively. Then $m(ABT)=0$ for all A and B in $\mathfrak{A}$. Since $BT \in L_1(\mathfrak{H}, \overline{\mathfrak{A}}, m)$, it follows from [10, Theorem 14], that $m(SBT)=0$ for all $S \in \overline{\mathfrak{A}}$, and therefore $BT(I-E_0)=0$ where E_0 is the maximal null projection. Letting $B \to I$ strongly, we have $T(I-E_0)=0$.

Corollary 1.3. *If $\mathfrak{A}$ is a self-adjoint operator algebra which is strongly dense in a W^*-algebra $\overline{\mathfrak{A}}$, then there is at most one gage m on $\overline{\mathfrak{A}}$ which makes $\mathfrak{A} \subset L_1(\mathfrak{H}, \overline{\mathfrak{A}}, m)$ and which has specified values on $\mathfrak{A}$.*

Proof. If m and m' were two such gages, apply Lemma 1.2 to

$$L_1(\mathfrak{H}, \overline{\mathfrak{A}}, m + m').$$

Corollary 1.4. *Let Γ be a locally compact Hausdorff space, and let $\mathfrak{A}$ be an algebra, closed under complex conjugation, of continuous functions vanishing at ∞ on Γ which separates points. Suppose μ and μ' are regular measures on Γ such that $\mathfrak{A} \subset L_1(\Gamma, \mu)$ and $\mathfrak{A} \subset L_1(\Gamma, \mu')$. If $\int f d\mu = \int f d\mu'$ for all $f \in \mathfrak{A}$, then $\mu = \mu'$.*

Proof. Apply Corollary 1.3 to the algebra of all multiplications by functions in $\mathfrak{A}$, acting on $L_2(\Gamma, \mu+\mu')$.

2. **Noncentral positive linear functionals.** The previous section suggests the question of what can be said about the extension of noncentral positive linear functionals. Theorem 2.1 characterizes the indefinite integral of a positive measurable operator for the simplest case of the W^*-algebra of all bounded operators.

Theorem 2.1. *Let $\mathfrak{H}$ be a Hilbert space and let l be a linear functional on the algebra $\mathfrak{F}$ of all bounded linear operators on $\mathfrak{H}$ with a finite dimensional range. Assume that l is non-negative in the sense that $l(T) \geqq 0$ if $T \geqq 0$ where $T \in \mathfrak{F}$. Suppose that there exists an orthonormal basis $\{e_\nu\}_{\nu \in N}$ of $\mathfrak{H}$ such that for every T in $\mathfrak{F}$*

$$\lim_F l(Q_F T Q_F) = l(T)$$

as F ranges over the finite subsets of N, directed by inclusion, where Q_F is projection on the linear subspace spanned by $\{e_\nu : \nu \in F\}$. Then there exists a bounded linear operator $A \geqq 0$ such that $l(T) = \mathrm{tr}(AT)$ for all T in $\mathfrak{F}$.

Proof. Let $\mathfrak{N} = \{T \in \mathfrak{F} : l(T^*T)=0\}$. It is easy to see (cf. Proof of Theorem 1.1) that $\mathfrak{N}$ is a left ideal in the algebra $\mathfrak{F}$, and hence $\mathfrak{F}/\mathfrak{N}$ is a left $\mathfrak{F}$-module. Also $\mathfrak{F}$ is a pre-Hilbert space under the inner product $(S+\mathfrak{N}, T+\mathfrak{N}) = l(T^*S)$. Let $\mathfrak{K}$ be the completion of $\mathfrak{F}/\mathfrak{N}$. Inequality (1) in the Proof of Theorem 1.1 also holds here and shows that $\mathfrak{F}$ acts as bounded operators on $\mathfrak{F}/\mathfrak{N}$. Therefore, its action can be extended to operate on $\mathfrak{K}$; the result is a norm-decreasing (inequality (1)) $*$-representation ρ of $\mathfrak{F}$ on $\mathcal{K}$. Let the canonical map of $\mathfrak{F}$ onto $\mathfrak{F}/\mathfrak{N}$ be denoted by $T\rightarrow\xi(T)$.

Let P_ν be projection on the one-dimensional subspace spanned by e_ν. Then for any $T\in\mathfrak{F}$

$$l(T) = \lim_F l(Q_FTQ_F) = \lim_F \sum_{\mu,\nu\in F} l(P_\mu TP_\nu) = \lim_F \sum_{\mu,\nu\in F} t_{\mu\nu}l(E_{\mu\nu})$$

where $(t_{\mu\nu})$ is the matrix of T relative to the basis $\{e_\nu\}$ and $E_{\mu\nu}$ is the partial isometry with one-dimensional initial space spanned by e_ν and one-dimensional final space spanned by e_μ. But, letting α be a fixed element of N, we have

$$\begin{aligned} l(E_{\mu\nu}) &= l(P_\mu E_{\mu\alpha}E_{\alpha\nu}P_\nu) \\ &= (\rho(E_{\alpha\nu})\xi(P_\nu),\ \rho(E_{\alpha\mu})\xi(P_\mu)) \\ &= (\zeta_\nu, \zeta_\mu) = a(\nu, \mu) \end{aligned}$$

where we have set $\zeta_\nu=\rho(E_{\alpha\nu})\xi(P_\nu)$ and the last equality defines $a(\nu, \mu)$.

Take T to be a one-dimensional projection, with range spanned, say, by $\sum_{\nu\in N}\lambda_\nu e_\nu$ with $\sum_{\nu\in N}|\lambda_\nu|^2=1$. Then $t_{\mu\nu}=\lambda_\mu\bar{\lambda}_\nu$ and so

$$l(T) = \lim_F \sum_{\mu,\nu\in F} \lambda_\mu\bar{\lambda}_\nu(\zeta_\nu, \zeta_\mu).$$

Hence we obtain

$$l(T) = \lim_F \left\| \sum_{\nu\in F} \bar{\lambda}_\nu\zeta_\nu \right\|^2. \tag{2}$$

Using Equation (2), we show that $\{\|\zeta_\nu\| : \nu\in N\}$ is bounded. For given $|\lambda_\nu|$, $\nu\in N$, we can arrange, by changing certain λ_ν to $-\lambda_\nu$ if necessary, that the inequality

$$\sum_{\nu\in F} |\lambda_\nu|^2\|\zeta_\nu\|^2 \leqq \left\| \sum_{\nu\in F} \lambda_\nu\zeta_\nu \right\|^2 \tag{3}$$

shall hold for all finite subsets F belonging to a certain cofinal family. To see this, we note that only a countable number of the λ_ν can differ from 0, say $\lambda_{\nu_1}, \lambda_{\nu_2}, \cdots$. The equation

$$\|\xi + \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 + 2\ \mathrm{Re}\ (\xi, \eta)$$

shows that either

$$\|\xi + \eta\|^2 \geqq \|\xi\|^2 + \|\eta\|^2$$

or else

$$\|\xi - \eta\|^2 \geqq \|\xi\|^2 + \|\eta\|^2.$$

Hence, proceeding by induction, we change λ_{ν_k} to $-\lambda_{\nu_k}$, if necessary, to achieve

$$\|\lambda_{\nu_1}\xi_{\nu_1} + \cdots + \lambda_{\nu_k}\xi_{\nu_k}\|^2 \geqq \|\lambda_{\nu_1}\xi_{\nu_1} + \cdots + \lambda_{\nu_{k-1}}\xi_{\nu_{k-1}}\|^2 + |\lambda_{\nu_k}|^2\|\xi_{\nu_k}\|^2$$
$$= |\lambda_{\nu_1}|^2\|\xi_{\nu_1}\|^2 + \cdots + |\lambda_{\nu_k}|^2\|\xi_{\nu_k}\|^2,$$

the last inequality arising from the inductive hypothesis. This proves that (3) holds whenever F intersects the sequence $\{\nu_k\}$ in an initial segment. Because (3) holds for a cofinal set of F's, we conclude that

$$\sum_{\nu\in N} |\lambda_\nu|^2\|\zeta_\nu\|^2 \leqq l(T) < +\infty.$$

But the above inequality holds for $|\lambda_\nu|^2$ arbitrary, subject to the condition $\sum_{\nu\in N}|\lambda_\nu|^2=1$. It follows that (cf. [1, Chapter V, §4]; also [14, p. 137, Theorem 3]) the function $\nu\to\|\zeta_\nu\|^2$ must be bounded.

Next, we use Equation (2) to show that for fixed $\mu\in N$, $\sum_{\nu\in N}|a(\mu,\nu)|^2 <+\infty$. Take λ_ν so that $a(\mu,\nu)\lambda_\nu\geqq 0$. Then for finite sets $F\subset F'$,

$$\sum_{\nu\in F} |a(\mu,\nu)|\,|\lambda_\nu| \leqq \sum_{\nu\in F'} a(\mu,\nu)\lambda_\nu = \left(\zeta_\mu, \sum_{\nu\in F'}\bar\lambda_\nu\zeta_\nu\right)$$
$$\leqq \|\zeta_\mu\|\cdot\left\|\sum_{\nu\in F'}\bar\lambda_\nu\zeta_\nu\right\|.$$

Since the above holds for all finite sets $F\subset F'$, it follows from Equation (2) that

$$\sum_{\nu\in N}|a(\mu,\nu)|\,|\lambda_\nu| \leqq \left[\sup_{\mu\in N}\|\zeta_\mu\|\right](l(T))^{1/2}. \tag{4}$$

Inequality (4) holds for arbitrary $|\lambda_\nu|$ subject to the condition $\sum_{\nu\in N}|\lambda_\nu|^2 =1$. It follows that (cf. [1, Chapter V, §4]; also [14, p. 137, Theorem 3])

$$\sum_{\nu\in N}|a(\mu,\nu)|^2 < +\infty.$$

The inequality (4) says that $\{a(\mu,\cdot):\mu\in N\}$ is a weakly bounded set of functions in $L_2(N)$. The Banach-Steinhaus Theorem [14, Theorem 1, p. 135] then tells us that the same collection is strongly bounded; that is to say, there exists a number K such that

$$\sum_{\nu\in N}|a(\mu,\nu)|^2 < K \qquad \text{for all } \mu\in N.$$

But this means that the operator A with matrix $(a(\mu,\nu))$ relative to the basis $\{e_\nu\}$ is everywhere defined on $\mathfrak{H}$ and is bounded by K. Hence for all $T\in\mathfrak{T}$

$$l(T) = \lim_F \sum_{\mu,\nu\in F} t_{\mu\nu}a(\nu,\mu) = \mathrm{tr}\,(TA).$$

II. Convergence in gage spaces

3. **Convergence in measure.** In [10, Def. 2.3], Segal introduced the notion of convergence nearly everywhere (n.e.) for a sequence of operators measurable with respect to a gage space. We include a definition here for convenience.

Definition. Let $(\mathcal{H}, \mathcal{A}, m)$ be a gage space. We say that a sequence $\{A_n\}$ of measurable operators *converges* n.e. to a measurable operator A if given $\epsilon>0$, there exists a nondecreasing sequence $\{P_n\}$ of projections in $\mathcal{A}$ such that $\|(A_n-A)P_n\|_\infty<\epsilon$, and $I-P_n$ is ultimately algebraically finite [10, §1.2, p. 404] and converges (strongly) to 0.

Remark. The following condition is sufficient for a sequence $\{A_n\}$ of measurable operators to converge n.e. to a measurable operator A: viz., there exists a sequence of projections $P_n\nearrow I$ such that $I-P_n$ is algebraically finite and $\|(A_n-A)P_n\|_\infty\to 0$.

Another important kind of convergence for functions on a measure space is convergence in measure [4, §22, p. 91]. We introduce a definition for measurable operators which is a straight imitation of the ordinary definition.

Definition. Let $(\mathcal{H}, \mathcal{A}, m)$ be a gage space. We say that a sequence $\{A_n\}$ of measurable operators *converges in measure* to a measurable operator A if given $\epsilon>0$, there exists a sequence $\{P_n\}$ of projections in $\mathcal{A}$ such that $\|(A_n-A)P_n\|_\infty<\epsilon$ and $m(I-P_n)\to 0$.

We prove here two statements to indicate that convergence in measure has appropriate connections with other concepts. More such propositions will be found in §5, which is devoted to the case $m(I)=1$.

Theorem 3.1. *Let $(\mathcal{H}, \mathcal{A}, m)$ be a gage space with a nonsingular gage (i.e., if P is a projection with $m(P)=0$, then $P=0$). If a sequence $\{A_n\}$ of measurable operators converges in measure to a measurable operator A, then some subsequence converges n.e. to A.*

Proof. We may assume without loss of generality that $A=0$. Let $\{\epsilon_n\}$ be a sequence of positive real numbers approaching 0. For each $r=1, 2, 3, \cdots$, choose a subsequence $n_{1r}, n_{2r}, n_{3r}, \cdots$, of the positive integers such that there exists projections Q_{jr} with $\|A_{n_{jr}}Q_{jr}\|_\infty<\epsilon_r$ and $m(I-Q_{jr})<2^{-j}$. We may also assume that the n_{jr} have been chosen increasing in r as well as in j. Set $P_{jr}=\bigcup_{k=j}^\infty Q_{kr}$. Then we have $\|A_{n_{jr}}P_{jr}\|<\epsilon_r$ and

$$m(I - P_{jr}) = m\left(\bigcup_{k=j}^{\infty} [I - Q_{kr}]\right) < \sum_{k=j}^{\infty} 2^{-k} = 2^{-j+1}.$$

Next, set $P_r=\bigcap_{k=r}^{\infty} P_{rr}$. Then $\|A_{n_{rr}}P_r\|<\epsilon_r$ and

$$m(I - P_r) = m\left(\bigcup_{k=r}^{\infty} [I - P_{kk}]\right) < \sum_{k=r}^{\infty} 2^{-k+1} = 2^{-r+2}.$$

Since m is a nonsingular gage, it follows that $P_r \nearrow I$ and $I-P_r$ is algebraically finite. The subsequence $\{A_{n_{rr}}\}$ satisfies the criterion of the remark.

THEOREM 3.2. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. If a sequence $\{A_n\}$ of operators in $L_p(\mathfrak{H}, \mathfrak{A}, m)$, $p=1$ or 2, converges in the sense of L_p to an operator A in $L_p(\mathfrak{H}, \mathfrak{A}, m)$, then $A_n \rightarrow A$ in measure.*

Proof. We may assume without loss of generality that $A=0$. Let $|A_n| = \int_0^\infty \lambda dE_\lambda^{(n)}$ be the spectral resolution of $|A_n|$. Then

$$\epsilon^p m(I - E_\epsilon^{(n)}) \leqq \int_0^\infty \lambda^p dm(E_\lambda^{(n)}) = \|A_n\|_p^p \rightarrow 0.$$

So $m(I-E_\epsilon^{(n)}) \rightarrow 0$ for all $\epsilon>0$, and

$$\|A_n E_\epsilon^{(n)}\|_\infty = \|\,|A_n|\, E_\epsilon^{(n)}\|_\infty \leqq \epsilon.$$

The next three theorems discuss the behavior of convergence in measure with respect to the various algebraic operations.

THEOREM 3.3. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. If two sequences $\{A_n\}$ and $\{B_n\}$ of measurable operators converge in measure to measurable operators A and B respectively, then $A_n+B_n \rightarrow A+B$ in measure.*

Proof. We may assume without loss of generality that $A=B=0$. Given $\epsilon>0$, choose sequences $\{P_n\}$ and $\{Q_n\}$ of projections in $\mathfrak{A}$ such that $\|A_nP_n\|_\infty < \epsilon$, $\|B_nQ_n\|_\infty < \epsilon$, $m(I-P_n) \rightarrow 0$, and $m(I-Q_n) \rightarrow 0$. Then $\|(A_n+B_n)(P_n \cap Q_n\|_\infty < 2\epsilon$ and

$$\begin{aligned} m(I - [P_n \cap Q_n]) &= m([I - P_n] \cup [I - Q_n]) \\ &\leqq m(I - P_n) + m(I - Q_n) \rightarrow 0. \end{aligned}$$

LEMMA 3.4. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. If $\mathfrak{N}$ is a closed subspace of $\mathfrak{H}$ affiliated with $\mathfrak{A}$ and S is essentially measurable, then*

$$[S^{-1}(\mathfrak{N})]^\perp = [S^*(\mathfrak{N}^\perp)]^-.$$

Proof I. We show that $S^*(\mathfrak{N}^\perp) \subset [S^{-1}(\mathfrak{N})]^\perp$. For let $x \in S^{-1}(\mathfrak{N})$ and $y \in \mathfrak{N}^\perp$. Then $(x, S^*y) = (Sx, y) = 0$ since $Sx \in \mathfrak{N}$.

II. Conversely, we show that $[S^{-1}(\mathfrak{N})]^\perp \subset [S^*(\mathfrak{N}^\perp)]^-$; or, what is the same thing, that $[S^*(\mathfrak{N}^\perp)]^\perp \subset [S^{-1}(\mathfrak{N})]^-$. But since $S^{-1}(\mathfrak{H})$ is strongly dense [10, Definition 2.1], $[S^*(\mathfrak{N}^\perp)]^\perp \cap S^{-1}(\mathfrak{H})$ is dense in $[S^*(\mathfrak{N}^\perp)]^\perp$; therefore, it is sufficient to show that

$$[S^*(\mathfrak{N}^\perp)]^\perp \cap S^{-1}(\mathfrak{H}) \subset S^{-1}(\mathfrak{N}).$$

Let $x \in [S^*(\mathfrak{N}^\perp)]^\perp \cap S^{-1}(\mathfrak{H})$. We show that $Sx \in \mathfrak{N}$ by showing that $Sx \perp \mathfrak{N}^\perp$. But again, since $(S^*)^{-1}(\mathfrak{H})$ is strongly dense, $\mathfrak{N}^\perp \cap (S^*)^{-1}(\mathfrak{H})$ is dense in $\mathfrak{N}^\perp$; therefore, it is sufficient to prove $Sx \perp \mathfrak{N}^\perp \cap (S^*)^{-1}(\mathfrak{H})$. Let $y \in \mathfrak{N}^\perp \cap (S^*)^{-1}(\mathfrak{H})$. Then

$$(Sx, y) = (x, S^*y) = 0 \qquad \text{since } x \in [S^*(\mathfrak{N}^\perp)]^\perp.$$

The next lemma is a strengthening of [10, Lemma 3.1]. Its proof is a slight modification of the proof in [10].

LEMMA 3.5. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. If $\mathfrak{N}$ is a closed subspace of $\mathfrak{H}$ affiliated with $\mathfrak{A}$, and S is a measurable operator, then $[S^{-1}(\mathfrak{N})]^\perp \lesssim \mathfrak{N}^\perp$. (For notation, see* [10, pp. 304, 305]). *Then, a fortiori,*

$$m\ (\text{proj. } on\ [S^{-1}(\mathfrak{N})]^\perp) \leqq m\ (\text{proj. } on\ \mathfrak{N}^\perp).$$

Proof. Let P be projection on $\mathfrak{N}^\perp$. Then by Lemma 3.4

$$[S^{-1}(\mathfrak{N})]^\perp = [S^*(\mathfrak{N}^\perp)]^- = [S^*P(\mathfrak{H})]^- \sim [PS(\mathfrak{H})]^- \subset \mathfrak{N}^\perp.$$

THEOREM 3.6. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. If a sequence $\{A_n\}$ of measurable operators converges in measure to a measurable operator A, then $A_n^* \to A^*$ in measure.*

Proof. We may assume without loss of generality that $A=0$. Let $A_n = U_n|A_n|$ be the polar decomposition of A_n. Given ϵ, choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $\|A_nP_n\|_\infty < \epsilon$ and $m(I-P_n)\to 0$. Let Q_n be projection on $[(U_n^*)^{-1}(P_n\mathfrak{H})]^-$. If $x\in (Q_n\mathfrak{H})\cap(A_n^*)^{-1}(\mathfrak{H})$, then

$$\|A_n^*x\| = \|\,|A_n|U_n^*x\| \leqq \epsilon\|U_n^*x\| \leqq \epsilon\|x\|.$$

Hence $\|A_n^*Q_n\|_\infty \leqq \epsilon$; and $m(I-Q_n)\leqq m(I-P_n)$ by Lemma 3.5 and so

$$m(I - Q_n) \to 0.$$

REMARK. We note that the following condition, which is analogous to another common definition of convergence in measure in ordinary measure theory, is equivalent to the definition of "$A_n \to A$ in measure" which has been given. Namely, given $\epsilon > 0$, there exists N such that for all $n > N$, there exists a projection P_n such that $\|(A_n - A)P_n\|_\infty < \epsilon$ and $m(I-P_n) < \epsilon$.

THEOREM 3.7. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. Let $\{A_n\}$ and $\{B_n\}$ be two sequences of measurable operators which converge in measure to measurable operators A and B respectively. Suppose that A and B have the following property: there exists projections P and Q in $\mathfrak{A}$ such that $m(I-P)$ and $m(I-Q)$ are finite and AP and BQ are bounded. Then $A_nB_n \to AB$ in measure.*

Proof. CASE I. $A=B=0$. Given $\epsilon>0$, choose sequences $\{P_n\}$ and $\{Q_n\}$ such that $\|A_nP_n\|_\infty<\epsilon$, $\|B_nQ_n\|_\infty<\epsilon$, $m(I-P_n)\to 0$, and $m(I-Q_n)\to 0$. Let R_n be projection on $[B_n^{-1}(P_n\mathfrak{H})]^-$. Then $m(I-R_n)\leqq m(I-P_n)$ by Lemma 3.5. Now

$$\|A_nB_n(R_n \cap Q_n)\|_\infty < \epsilon^2$$

while

$$\begin{aligned} m(I - [R_n \cap Q_n]) &= m([I - R_n] \cup [I - Q_n]) \\ &= m(I - R_n) + m(I - Q_n) \to 0. \end{aligned}$$

CASE II. $A_n = A$, $B = 0$. We use the above remark to show that $AB_n \to 0$ in measure. Let $|A(I-P)| = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of $|A(I-P)|$. Given $\epsilon > 0$, choose $\Lambda > \|AP\|_\infty$ so that $m(I-E_\Lambda) < \epsilon$. Let N be such that for $n > N$, there exists a projection Q_n with $\|B_nQ_n\|_\infty < \epsilon/\Lambda$ and $m(I-Q_n) < \epsilon$. Let R_n be projection on $[B_n^{-1}(E_\Lambda \mathcal{H})]^-$. Then $m(I-R_n) \leqq m(I-E_\Lambda) < \epsilon$ by Lemma 3.5. If $x \in (R_n \cap Q_n)\mathcal{H}$, then

$$\begin{aligned} \|AB_nx\| &\leqq \|A(I-P)B_nx\| + \|APB_nx\| \\ &= \| \, |A(I-P)| \, B_nx\| + \|APB_nx\| \\ &\leqq 2\Lambda \|B_nx\| \leqq 2\Lambda(\epsilon/\Lambda) = 2\epsilon. \end{aligned}$$

Hence $\|AB_n(R_n \cap Q_n)\|_\infty \leqq 2\epsilon$ while

$$\begin{aligned} m(I - [R_n \cap Q_n]) &= m([I - R_n] \cup [I - Q_n]) \\ &\leqq m(I - R_n) + m(I - Q_n) < 2\epsilon. \end{aligned}$$

CASE III. $A = 0$, $B_n = B$. By Case II and Theorem 3.6.

CASE IV. *General.* Let $A_n = A + A_n'$ and $B_n = B + B_n'$. Then $A_nB_n - AB = A_n'B + AB_n' + A_n'B_n'$. Hence Case IV reduces to Cases I, II, and III via Theorem 3.3.

4. **Dominated convergence theorems.** One of the most important tools in ordinary integration theory for interchanging limits with integrals is the Lebesgue dominated convergence theorem [4, Theorem 26.D]. The intractibility of absolute value and inequalities for operators makes various statements of the ordinary theorem inequivalent in the noncommutative case. For this reason several dominated convergence theorems are given below. Since these theorems hold both for convergence n.e. and for convergence in measure, it is convenient to formulate a mode of convergence which is implied by both of these.

DEFINITION. Let $(\mathcal{H}, \mathcal{A}, m)$ be a gage space and let $T \in L_1(\mathcal{H}, \mathcal{A}, m)$. We shall say that a projection P in $\mathcal{A}$ is *T-majorized* by a number ϵ if $|m(TQ)| \leqq \epsilon$ for any projection $Q \lesssim P$. We shall say that a sequence $\{P_n\}$ of projections in $\mathcal{A}$ is *T-infinitesimal* if given $\epsilon > 0$, there exists N such that P_n is T-majorized by ϵ for $n \geqq N$.

LEMMA 4.1. *Let $(\mathcal{H}, \mathcal{A}, m)$ be a gage space and let $T \in L_1(\mathcal{H}, \mathcal{A}, m)$. If projections P and Q in $\mathcal{A}$ are T-majorized by ϵ, then $P \cup Q$ is T-majorized by 2ϵ.*

Proof. Suppose $R \lesssim P \cup Q$. Then R can be decomposed into disjoint projections, $R = R' + R''$, so that $R' \lesssim P$ and $R'' \lesssim Q$. Then

$$|m(TR)| \leqq |m(TR')| + |m(TR'')| < 2\epsilon.$$

DEFINITION. Let $(\mathcal{H}, \mathcal{A}, m)$ be a gage space. We shall say that a sequence $\{A_n\}$ of measurable operators *converges grossly* to a measurable operator A if for every $T \in L_1(\mathcal{H}, \mathcal{A}, m)$ and every $\epsilon > 0$, there exists a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $\|(A_n-A)P_n\|_\infty<\epsilon$ and $\{I-P_n\}$ is T-infinitesimal.

REMARK. It is clear that $A_n\to A$ grossly if and only if for every $T\in L_1(\mathfrak{H},\mathfrak{A},m)$ and every $\epsilon>0$, there exists N such that for all $n\geqq N$, there exists a projection P_n such that $\|(A_n-A)P_n\|_\infty<\epsilon$ and $I-P_n$ is T-majorized by ϵ. It is also clear that in either formulation of gross convergence, it is sufficient to consider $T\geqq 0$.

LEMMA 4.2. *Let $(\mathfrak{H},\mathfrak{A},m)$ be a gage space. If a sequence $\{A_n\}$ of measurable operators converges to a measurable operator A either n.e. or in measure, then $A_n\to A$ grossly.*

Proof. We may assume without loss of generality that $A=0$. If $A_n\to 0$ in measure, then given $\epsilon>0$, we can choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $\|A_nP_n\|_\infty<\epsilon$ and $m(I-P_n)\to 0$. Then $\{I-P_n\}$ is T-infinitesimal for any $T\in L_1(\mathfrak{H},\mathfrak{A},m)$ by Lemma 4.3 below.

If, on the other hand, $A_n\to 0$ n.e., then given $\epsilon>0$, we can choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $\|A_nP_n\|_\infty<\epsilon$, $I-P_n\searrow 0$, and $I-P_n$ is algebraically finite for $n\geqq N$. It follows then from Lemma 4.4 below that $\{I-P_n\}$ is T-infinitesimal.

LEMMA 4.3. *Let $(\mathfrak{H},\mathfrak{A},m)$ be a gage space and let $T\in L_1(\mathfrak{H},\mathfrak{A},m)$. Given $\epsilon>0$, there exists $\delta>0$ such that for every projection $P\in\mathfrak{A}$ with $m(P)<\delta$, $|m(TP)|<\epsilon$.*

Proof. We may assume without loss of generality that $T\geqq 0$. If the lemma were false, for some $\epsilon>0$, we could choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $m(P_n)<2^{-n}$ but $m(TP_n)\geqq\epsilon$. Set $Q_n=\bigcup_{k=n}^\infty P_k$. Then $m(Q_n)<\sum_{k=n}^\infty 2^{-k}=2^{-n+1}$. Since m is completely additive, it follows that Q_n decreases to the maximal null projection. However $m(TQ_n)\geqq m(TP_n)\geqq\epsilon$, and this contradicts the strong continuity of the extended indefinite integral of T on the unit ball of $\mathfrak{A}$ [10, Theorem 14].

LEMMA 4.4. *Let $(\mathfrak{H},\mathfrak{A},m)$ be a gage space. If $\{P_n\}$ is a sequence of algebraically finite projections $\searrow 0$, then $\{P_n\}$ is T-infinitesimal for all*

$$T\in L_1(\mathfrak{H},\mathfrak{A},m).$$

Proof. We may suppose without loss of generality that $T\geqq 0$, that m is nonsingular, and that $\mathfrak{A}$ has a countably decomposable center [10, Lemma 11.1]. Let d be a dimension function whose values are functions on a finite measure space [10, Theorem 1]. If the lemma were false, there would exist a subsequence $\{k(n)\}$ of the integers, a sequence of projections $\{R_n\}$ and a number $\epsilon>0$, such that $R_n\lesssim P_{k(n)}$, but $m(TR_n)>\epsilon$. Let $f_n=d(P_{k(n)})$. Applying Lemma 6.1 in [10] to $f_{ij}=f_j$, we obtain a subsequence $\{f_{j(n)}\}$ such that $\sum_{n=1}^\infty f_{j(n)}<+\infty$ n.e. Set $Q_n=\bigcup_{l=n}^\infty R_{j(l)}$. Then $d(Q_n)\leqq\sum_{l=n}^\infty f_{j(l)}\to 0$ n.e., and since d is countably additive, Q_n decreases to 0. As in the proof of Lemma 4.3, this is impossible.

Gross convergence behaves better relative to the algebraic operations than does either convergence n.e. or convergence in measure.

LEMMA 4.5. *Let* $(\mathcal{H}, \mathfrak{A}, m)$ *be a gage space. Let* $\{A_n\}$ *and* $\{B_n\}$ *be two sequences of measurable operators which converge grossly to measurable operators* A *and* B *respectively. Then* $A_n+B_n\to A+B$ *grossly,* $A_n^*\to A^*$ *grossly, and* $A_nB_n\to AB$ *grossly.*

Proof. In proving that $A_n+B_n\to A+B$ grossly and that $A_n^*\to A^*$ grossly, we may assume without loss of generality that $A=B=0$. Let $T\in L_1(\mathcal{H}, \mathfrak{A}, m)$. To show that $A_n+B_n\to 0$ grossly, given $\epsilon>0$, we choose sequences $\{P_n\}$ and $\{Q_n\}$ of projections in $\mathfrak{A}$ such that $\|A_nP_n\|_\infty<\epsilon$, $\|B_nQ_n\|_\infty<\epsilon$, and $\{I-P_n\}$ and $\{I-Q_n\}$ are T-infinitesimal. Then

$$\|(A_n+B_n)(P_n\cap Q_n)\|_\infty<2\epsilon \text{ and } \{I-(P_n\cap Q_n)\}=\{(I-P_n)\cup(I-Q_n)\}$$

is T-infinitesimal by Lemma 4.1. In order to show that $A_n^*\to 0$ grossly, given $\epsilon>0$, we choose $\{P_n\}$ as above. Let Q_n be defined from P_n and A_n as in the proof of Theorem 3.6. Then $\|A_n^*Q_n\|_\infty<\epsilon$ and $\{I-Q_n\}$ is T-infinitesimal since $I-Q_n\lesssim I-P_n$.

To prove that $A_nB_n\to AB$ grossly, we consider cases as in the proof of Theorem 3.7.

CASE I. $A=B=0$. Given $\epsilon>0$, choose sequences $\{P_n\}$ and $\{Q_n\}$ such that $\|A_nP_n\|_\infty<\epsilon$, $\|B_nQ_n\|_\infty<\epsilon$, and $\{I-P_n\}$ and $\{I-Q_n\}$ are T-infinitesimal. Define R_n as in case I of the proof of Theorem 3.7. Then $\{I-R_n\}$ is T-infinitesimal since $I-R_n\lesssim I-Q_n$. Now $\|A_nB_n(R_n\cap Q_n)\|_\infty<\epsilon$ and $\{I-(R_n\cap Q_n)\}=\{(I-R_n)\cup(I-Q_n)\}$ is T-infinitesimal by Lemma 4.1.

CASE II. $A_n=A$, $B=0$. Since A is measurable, there exists a sequence $\{P_n\}$ of projections such that $\|AP_n\|_\infty<+\infty$, $I-P_n\searrow 0$ and $I-P_n$ is algebraically finite for $n\geqq N'$. By Lemma 4.4, $\{I-P_n\}$ is T-infinitesimal. Given $\epsilon>0$, choose N'' so that $I-P_{N''}$ is T-majorized by ϵ, and let $\Lambda=\|AP_{N''}\|_\infty$. Let $N\geqq N''$ be such that for $n\geqq N$, there exists a projection Q_n with $\|B_nQ_n\|_\infty<\epsilon/\Lambda$ and $I-Q_n$ T-majorized by ϵ. Let R_n be projection on $[B_n^{-1}(P_{N''}\mathcal{H})]^-$. Then $I-R_n\lesssim I-P_{N''}$ and so $I-R_n$ is T-majorized by ϵ. Then $\|AB_n(R_n\cap Q_n)\|_\infty<\Lambda\cdot\epsilon/\Lambda=\epsilon$ and $I-(R_n\cap Q_n)=(I-R_n)\cup(I-Q_n)$ is T-majorized by 2ϵ for $n\geqq N$, by Lemma 4.1.

CASE III. $A=0$, $B_n=B$. As in Case III of Theorem 3.7.

CASE IV. *General.* As in Case IV of Theorem 3.7.

THEOREM 4.6. *Let* $(\mathcal{H}, \mathfrak{A}, m)$ *be a gage space. Let* $\{A_n\}$ *be a sequence of measurable operators which converges grossly to a measurable operator* A. *Suppose there exists a non-negative integrable operator* B *such that* $|A_n-A|\leqq B$ *for* $n=1, 2, 3, \cdots$. *Then* $A_n-A\to 0$ *in* $L_1(\mathcal{H}, \mathfrak{A}, m)$.

Proof. We may assume without loss of generality that $A=0$. Let $B=\int_0^\infty\lambda dE_\lambda$ be the spectral resolution of B. Given $\epsilon>0$, choose $\delta>0$ so that

$m(BE_\delta) < \epsilon$. Choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that

$$\|A_n P_n\|_\infty = \| \, |A_n| \, P_n\|_\infty < \frac{\epsilon\delta}{m(B)}$$

and $\{I - P_n\}$ is B-infinitesimal. Set $Q_n = E_\delta \cup (I - P_n)$; then $I - Q_n = (I - E_\delta) \cap P_n$. Now

$$\|A_n\|_1 = m(|A_n| Q_n) + m(|A_n| (I - Q_n))$$
$$\leqq m(BQ_n) + \frac{\epsilon\delta}{m(B)} m(I - E_\delta).$$

But $Q_n - E_\delta \precsim I - P_n$ and therefore $m(BQ_n) \to m(BE_\delta)$; also $\delta m(I - E_\delta) \leqq m(B)$. Hence

$$\|A_n\|_1 \leqq m(BQ_n) + \epsilon \to m(BE_\delta) + \epsilon < 2\epsilon.$$

COROLLARY 4.7. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a nonsingular gage space. If $\{A_n\}$ is a sequence of non-negative measurable operators which converges grossly to a measurable operator A, then A also is non-negative.*

Proof. Since A is measurable, it is sufficient to show that $m(AQ) \geqq 0$ whenever Q is a projection such that $m(Q) < +\infty$ and $AQ \in L_1(\mathfrak{H}, \mathfrak{A}, m)$. By Lemma 4.5, $(A_n - A)Q \to 0$ grossly. Hence there exists a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $\|(A_n - A)QP_n\|_\infty < 1$ and $\{I - P_n\}$ is Q-infinitesimal. In particular $m((I - P_n) \cap Q) = m(Q[(I - P_n) \cap Q]) \to 0$, and so $P_n \cup (I - Q) \to I$ in measure. Since

$$Q(P_n \cup (I - Q))\mathfrak{H} \subset QP_n\mathfrak{H}$$

it follows that

$$\|Q(A_n - A)Q[P_n \cup (I - Q)]\|_\infty < 1$$

and so $|[Q(A_n - A)Q(P_n \cup (I - Q))]^*| \leqq Q$. Therefore $QA_nQ(P_n \cup (I - Q)) \in L_1(\mathfrak{H}, \mathfrak{A}, m)$. By Theorem 4.6 and Lemma 4.5,

$$m(QA_nQ(P_n \cup (I - Q))) \to m(QAQ) = m(AQ).$$

But $m(QA_nQ(P_n \cup (I - Q))) \geqq 0$ and hence $m(AQ) \geqq 0$.

The bounding condition in Theorem 4.6 involves both A_n and A. If we assume that $|A_n| \leqq B$ and $|A| \leqq B$, it does not follow that $|A_n - A| \leqq 2B$ since the inequality $|S + T| \leqq |S| + |T|$ is not in general valid for operators. If we desire to have a dominated theorem in which the bounding condition involves only A_n, we may use the following:

THEOREM 4.8. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. Let $\{A_n\}$ be a sequence of measurable operators which converges grossly to a measurable operator A. Suppose there exists a non-negative integrable operator B such that $-B \leqq \mathrm{Re}\, A_n \leqq B$*

and $-B \leqq \operatorname{Im} A_n \leqq B$ *for* $n=1, 2, 3, \cdots$. *Then* $A \in L_1(\mathcal{H}, \mathcal{A}, m)$ *and* $A_n \to A$ *in* $L_1(\mathcal{H}, \mathcal{A}, m)$.

Proof. By Corollary 4.7, there is no loss of generality in assuming that $-B \leqq \operatorname{Re} A \leqq B$ and $-B \leqq \operatorname{Im} A \leqq B$. Hence we may as well assume $A=0$. By Lemma 4.5, we may suppose without loss of generality that A_n is self-adjoint. Given $\epsilon > 0$, we choose E_δ, P_n, and Q_n as in the proof of Theorem 4.6. Now

$$\begin{aligned} \|A_n\|_1 &\leqq \|Q_n A Q_n\|_1 + \|Q_n A_n(I - Q_n)\|_1 + \|(I - Q_n)A_n\|_1 \\ &\leqq \|Q_n A Q_n\|_1 + 2\|A_n(I - Q_n)\|_1 \end{aligned}$$

since

$$A_n(I - Q_n) = [(I - Q_n)A_n]^*.$$

But

$$\|A_n(I - Q_n)\|_1 < \frac{\epsilon\delta}{m(B)} m(I - E_\delta) < \epsilon.$$

As for $\|Q_n A Q_n\|_1$, we have

$$-Q_n B Q_n \leqq Q_n A Q_n \leqq Q_n B Q_n.$$

Let F^+ and F^- be spectral projections for $Q_n A Q_n$ such that $Q_n A Q_n F^+ \geqq 0$ and $Q_n A Q_n F^- \leqq 0$. Then

$$Q_n A Q_n F^+ = F^+ Q_n A Q_n F^+ \leqq F^+ Q_n B Q_n F^+$$

and

$$-Q_n A Q_n F^- = -F^- Q_n A Q_n F^- \leqq F^- Q_n B Q_n F^-.$$

Hence $|Q_n A Q_n| = Q_n A Q_n F^+ - Q_n A Q_n F^- \leqq F^+ Q_n B Q_n F^+ + F^- Q_n B Q_n F^-$ and so

$$\begin{aligned} \|Q_n A Q_n\|_1 &\leqq m(Q_n B Q_n F^+) + m(Q_n B Q_n F^-) \\ &\leqq m(Q_n B Q_n) = m(B Q_n) \to m(B E_\delta). \end{aligned}$$

Hence

$$\|A_n\|_1 = m(B Q_n) + 2\epsilon \to m(B E_\delta) + 2\epsilon < 3\epsilon.$$

The bounding condition appears more natural in the corresponding L_2 theorem.

THEOREM 4.9. *Let* $(\mathcal{H}, \mathcal{A}, m)$ *be a gage space. Let* $\{A_n\}$ *be a sequence of measurable operators which converges grossly to a measurable operator* A. *Suppose there exists a non-negative integrable operator* B *such that* $|A_n|^2 = A_n^* A_n \leqq B$ *for* $n = 1, 2, 3, \cdots$. *Then* $A \in L_2(\mathcal{H}, \mathcal{A}, m)$ *and* $A_n \to A$ *in* $L_2(\mathcal{H}, \mathcal{A}, m)$.

Proof. Let $B = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of B. Given $\epsilon > 0$, choose

$\delta > 0$ so that $m(BE_\delta) < \epsilon$. Choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that

$$\|(A_n - A)P_n\|_\infty < [\epsilon\delta/m(B)]^{1/2}$$

and $\{I - P_n\}$ is B infinitesimal. Set $Q_n = E_\delta \cup (I - P_n)$; then $I - Q_n = (I - E_\delta) \cap P_n$. Now

$$\|A_n - A\|_2^2 = m((A_n - A)^*(A_n - A)Q_n) + m((A_n - A)^*(A_n - A)(I - Q_n)).$$

But $m((A_n - A)^*(A_n - A)(I - Q_n)) < (\epsilon\delta/m(B))m(I - E_\delta) < \epsilon$. Also, assuming as we may without loss of generality by Corollary 4.7 and Lemma 4.5 that $A^*A \leqq B$, we have

$$(A_n - A)^*(A_n - A) = 2(A_n^*A_n + A^*A) - (A_n + A)^*(A_n + A) \leqq 4B,$$

and so $m((A_n - A)^*(A_n - A)Q_n) \leqq 4m(BQ_n) \to 4m(BE_\delta)$. Hence

$$\|A_n - A\|_2^2 < 4m(BQ_n) + \epsilon \to 4m(BE_\delta) + \epsilon < 5\epsilon.$$

Remark. It is sufficient to assume that $(\operatorname{Re} A_n)^2 \leqq B$ and $(\operatorname{Im} A_n)^2 \leqq B$ in Theorem 4.9 instead of $A_n^*A_n \leqq B$. For under these assumptions

$$\begin{aligned} A_n^*A_n &= (\operatorname{Re} A_n + i \operatorname{Im} A_n)^*(\operatorname{Re} A_n + i \operatorname{Im} A_n) \\ &\leqq 2[(\operatorname{Re} A_n)^2 + (\operatorname{Im} A_n)^2] \\ &\quad - (\operatorname{Re} A_n - i \operatorname{Im} A_n)^*(\operatorname{Re} A_n - i \operatorname{Im} A_n) \\ &\leqq 4B. \end{aligned}$$

We give now a noncommutative version of Fatou's lemma [4, Theorem 27.F]. We remark that the lemma is almost always applied when the sequence of functions actually converges, so that the following theorem does generalize the important case.

Theorem 4.10. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. If a sequence $\{A_n\}$ of nonnegative measurable operators converges grossly to a measurable operator A, then $m(A) \leqq \liminf_{n\to\infty} m(A_n)$.*

Proof. By Corollary 4.7, we may assume without loss of generality that $A \geqq 0$. It is sufficient to show that

$$m(AQ) \leqq L = \liminf_{n\to\infty} m(A_n)$$

whenever Q is a projection such that $m(Q) < +\infty$ and $AQ \in L_1(\mathfrak{H}, \mathfrak{A}, m)$. We repeat the argument of Corollary 4.7, which shows that

$$m(AQ) = \lim_{n\to\infty} m(AQ_nQ(P_n \cup (I - Q))) \leqq L.$$

5. **Convergence in measure in probability gage spaces.** By a probability gage space, we mean a gage space $(\mathfrak{H}, \mathfrak{A}, m)$ such that $m(I) = 1$. When $I \in L_1(\mathfrak{H}, \mathfrak{A}, m)$, it is clear that gross convergence implies convergence in

measure. Hence, by Lemma 4.2, gross convergence and convergence in measure are equivalent for probability gage spaces. It follows then from Lemma 4.2, that convergence n.e. for probability gage spaces implies convergence in measure.

For probability measure spaces convergence in measure is the same as convergence arising from a certain metric. We show that the same thing is true for probability gage spaces.

THEOREM 5.1. *Let* $(\mathcal{H}, \mathcal{A}, m)$ *be a probability gage space. A sequence* $\{A_n\}$ *of measurable operators converges in measure to a measurable operator* A *if and only if*

$$m(|A_n - A|(I + |A_n - A|)^{-1}) \to 0.$$

Proof. We may assume without loss of generality that $A=0$. Suppose $A_n \to 0$ in measure. Then given $\epsilon > 0$, we choose N so that for $n \geqq N$ there exists a projection P_n such that $\|A_n P_n\|_\infty < \epsilon$ and $m(I-P_n) < \epsilon$. Then

$$\begin{aligned} m(|A_n|(I + |A_n|)^{-1}) & \\ &= m(|A_n|(I + |A_n|)^{-1}P_n) + m(|A_n|(I + |A_n|)^{-1}(I - P_n)) \\ &\leqq m(|A_n|P_n) + m(I - P_n) < \||A_n|P_n\|_\infty + \epsilon \\ &= \|A_n P_n\|_\infty + \epsilon < 2\epsilon. \end{aligned}$$

Conversely, suppose that $m(|A_n|(I+|A_n|)^{-1}) \to 0$. Let $|A_n| = \int_0^\infty \lambda dE_\lambda^{(n)}$ be the spectral resolution of $|A_n|$. Given ϵ, choose N so large that for $n \geqq N$

$$m(|A_n|(I + |A_n|)^{-1}) < \epsilon^2.$$

Set $P_n = E_\epsilon^{(n)}$. Then $\|A_n P_n\|_\infty = \||A_n|P_n\|_\infty \leqq \epsilon$. Also

$$\epsilon(1 + \epsilon)^{-1} m(I - P_n) \leqq \int_\epsilon^\infty \lambda(1 + \lambda)^{-1} dm(E_\lambda^{(n)}) \leqq m(|A_n|(I + |A_n|)^{-1}) < \epsilon^2.$$

Hence $m(I-P_n) < \epsilon(1+\epsilon)$.

The proof of Theorem 5.1 makes the following corollary clear.

COROLLARY 5.2. *Let* $(\mathcal{H}, \mathcal{A}, m)$ *be a probability gage space. Let* $\{A_n\}$ *be a sequence of measurable operators and* A *a measurable operator. Let* $|A_n - A| = \int_0^\infty \lambda dE_\lambda^{(n)}$ *be the spectral resolution of* $|A_n - A|$. *Then* $A_n \to A$ *in measure if and only if* $m(I - E_\epsilon^{(n)}) \to 0$ *for all* $\epsilon > 0$.

We next investigate how convergence in measure for self-adjoint operators is related to convergence of continuous functions of the operators. The methods are drawn from [8, Lemmas 1.5.1–1.5.3].

LEMMA 5.3. *Let* $(\mathcal{H}, \mathcal{A}, m)$ *be a probability gage space. If a sequence* $\{A_n\}$ *of self-adjoint measurable operators converges in measure to a self-adjoint measurable operator* A, *then the sequence of Cayley transforms* $\{(A_n+iI)/(A_n-iI)\}$

converges to $(A+iI)/(A-iI)$ *in* $L_2(\mathcal{H}, \mathfrak{A}, m)$.

Proof. It is sufficient to show that $(A_n+iI)/(A_n-iI)\to(A+iI)/(A-iI)$ in measure; for since the Cayley transforms are unitary, L_2-convergence would then follow from the dominated convergence Theorem 4.9. Given $\epsilon>0$, choose a sequence $\{P_n\}$ of projections in $\mathfrak{A}$ such that $\|(A_n-A)P_n\|_\infty<\epsilon$ and $m(I-P_n)<\epsilon$ for $n\geqq N$. We use the equation, taken from [8],

$$\frac{A_n+iI}{A_n-iI}-\frac{A+iI}{A-iI}=-2i(A_n-iI)^{-1}(A_n-A)(A-iI)^{-1}.$$

Let Q_n be projection on $[(A-iI)P_n\mathcal{H}]^-$. Then the above equation shows that for $n\geqq N$

$$\left\|\left(\frac{A_n+iI}{A_n-iI}-\frac{A+iI}{A-iI}\right)Q_n\right\|_\infty<2\epsilon\|(A_n-iI)^{-1}\|_\infty\|(A-iI)^{-1}\|_\infty\leqq 2\epsilon.$$

Furthermore $(A-iI)^{-1}\in\mathfrak{A}$. Therefore, by Lemma 3.5,

$$m(I-Q_n)\leqq m(I-P_n)<\epsilon.$$

Corollary 5.4. *Let* $(\mathcal{H}, \mathfrak{A}, m)$ *be a probability gage space. If a sequence* $\{A_n\}$ *of self-adjoint measurable operators converges in measure to a self-adjoint measurable operator* A, *then* $\phi(A_n)\to\phi(A)$ *in* $L_2(\mathcal{H}, \mathfrak{A}, m)$ *where* ϕ *is any continuous function on the one-point compactification of the real line.*

Proof. Let $\phi(\lambda)=\psi((\lambda+i)(\lambda-i)^{-1})$ where ψ is a continuous function on the unit circle. By Lemma 5.3,

$$\frac{A_n+iI}{A_n-iI}\to\frac{A+iI}{A-iI}\quad\text{in } L_2(\mathcal{H}, \mathfrak{A}, m).$$

Then by the argument of [8, Lemma 1.5.2]

$$\phi(A_n)=\psi\left(\frac{A_n+iI}{A_n-iI}\right)\to\psi\left(\frac{A+iI}{A-iI}\right)=\phi(A)$$

in $L_2(\mathcal{H}, \mathfrak{A}, m)$.

Theorem 5.5. *Let* $(\mathcal{H}, \mathfrak{A}, m)$ *be a probability gage space. Let* $\{A_n\}$ *be a sequence of self-adjoint measurable operators and let* A *be a self-adjoint measurable operator. A necessary and sufficient condition that* $A_n\to A$ *in measure is that* $\phi(A_n)\to\phi(A)$ *in* $L_2(\mathcal{H}, \mathfrak{A}, m)$ *for all continuous functions* ϕ *with compact support on the real line.*

Proof. The necessity of the condition follows from Corollary 5.4. To prove sufficiency, suppose that the condition is fulfilled. Let $|A_n|=\int_0^\infty\lambda dE_\lambda^{(n)}$ and $|A|=\int_0^\infty\lambda dE_\lambda$ be the spectral resolutions of $|A_n|$ and $|A|$. Given $\epsilon>0$, choose Λ so that $m(I-E_\Lambda)<\epsilon$, where E_Λ means $\lim_{\lambda\nearrow\Lambda}E_\lambda$. Let ϕ be a continuous function with compact support such that $\phi(\lambda)=\lambda$ for $-\Lambda\leqq\lambda\leqq\Lambda$. Let $|\phi(A_n)-\phi(A)|=\int_0^\infty\lambda dF_\lambda^{(n)}$ be the spectral resolution of $|\phi(A_n)-\phi(A)|$. Then we have

$$\|(A_n - A)(E_\Lambda^{(n)} \cap E_\Lambda \cap F_\epsilon^{(n)})\|_\infty$$
$$= \|(\phi(A_n) - \phi(A))(E_\Lambda^{(n)} \cap E_\Lambda \cap F_\epsilon^{(n)})\|_\infty \leqq \epsilon,$$

and

$$m(I - (E_\Lambda^{(n)} \cap E_\Lambda \cap F_\epsilon^{(n)})) \leqq m(I - E_\Lambda^{(n)}) + m(I - E_\Lambda) + m(I - F_\epsilon^{(n)}).$$

Choose N' so that $\|\phi(A_n)-\phi(A)\|_2^2<\epsilon^3$ for $n\geqq N'$; then $\epsilon^2 m(I-F_\epsilon^{(n)}) \leqq\|\phi(A_n)-\phi(A)\|_2^2$ and so $m(I-F_\epsilon^{(n)})<\epsilon$. Also by choice of Λ, $m(I-E_\Lambda)<\epsilon$. Now the equation

$$\lim_{n\to\infty}\int_0^\infty f(\lambda)dm(E_\lambda^{(n)}) = \int_0^\infty f(\lambda)dm(E_\lambda)$$

must hold for a uniformly closed set of continuous functions f. By hypothesis, it holds when f has compact support, and it holds trivially when f is identically 1. Therefore, it holds for all continuous functions f with a finite limit at $+\infty$. Let f be $\geqq$ the characteristic function of $[\Lambda, +\infty)$, but such that

$$\int_0^\infty f(\lambda)dm(E_\lambda) < 2\epsilon.$$

Choose N'' so that for $n\geqq N''$, $\int_0^\infty f(\lambda)dm(E_\lambda^{(n)})<3\epsilon$. Then for $n\geqq\max(N', N'')$

$$m(I - (E_\Lambda^{(n)} \cap E_\Lambda \cap F_\epsilon^{(n)})) \leqq 2\epsilon + \int_0^\infty f(\lambda)dm(E_\lambda^{(n)}) < 5\epsilon.$$

6. **A monotone convergence theorem.** In [10], Segal gives a monotone convergence theorem (Corollary 13.2). In this theorem, the convergence is in the sense of n.e. We shall have occasion to use monotone convergence in a "pointwise" sense in connection with the Fubini theorem. This is most conveniently formulated in terms of "quadratic functionals."

DEFINITION. A *quadratic functional* on a Hilbert space $\mathcal{H}$ is a function q from $\mathcal{H}$ to $[0, +\infty]$ (including $+\infty$) such that for all x and y in $\mathcal{H}$ and all complex numbers λ,

1. $q(x+y)+q(x-y)=2q(x)+2q(y)$, and
2. $q(\lambda x)=|\lambda|^2q(x)$. (This will be interpreted to mean $q(0)=0$.)

For example, if T is any non-negative (not necessarily densely defined) linear operator on $\mathcal{H}$, then setting

$$q(x) = \begin{cases}(Tx, x) & \text{if } x \text{ is in the domain of } T,\\ +\infty & \text{otherwise}\end{cases}$$

makes q a quadratic functional. A second example is given in the following definition.

DEFINITION. Let T be a non-negative self-adjoint operator on a Hilbert space $\mathfrak{H}$. By the *standard quadratic functional corresponding to* T, denoted by q_T, we mean the quadratic functional defined as follows:

$$q_T(x) = \begin{cases} \| T^{1/2}x\|^2 & \text{if } x \text{ is in the domain of } T^{1/2}, \\ +\infty & \text{otherwise.} \end{cases}$$

The connection between quadratic functionals and bilinear functionals is given in the following lemma.

LEMMA 6.1. *Let q be a quadratic functional on a Hilbert space $\mathfrak{H}$. Then the set L of all x such that $q(x) < +\infty$ is a linear manifold. The result b of polarizing q on L:*

$$b(x, y) = \frac{1}{4}\left[q(x + y) - q(x - y) - iq(ix + y) + iq(ix - y)\right] \quad \textit{for } x \textit{ and } y \textit{ in } L$$

is a non-negative Hermitian symmetric bilinear functional on $L \times L$ and $q(x) = b(x, x)$ for all x in L.

Proof. The fact that L is linear is obvious from the definition of quadratic functional. The argument in [5] shows that b is bilinear over the field K of all $r+si$ where r and s are rational. Furthermore, the argument in [5] shows that $b(x, x) = q(x)$ for any x in L. The definition of b makes it clear that $b(x, y) = [b(y, x)]^-$, for any x and y in L. Hence

$$|\lambda|^2 b(x, x) + 2 \operatorname{Re}[\lambda\bar{\mu} b(x, y)] + |\mu|^2 b(y, y) = b(\lambda x + \mu y, \lambda x + \mu y)$$

at least for λ and μ in K. Therefore, the quadratic form in λ and μ is positive definite, and from this we get the Schwarz inequality:

$$|b(x, y)|^2 \leqq b(x, x) b(y, y) = q(x) q(y).$$

But then the homogeneity property of q (property 2 in the definition of quadratic functional) implies that $b(\lambda x, y)$ is continuous in λ. It follows that b is bilinear over the complex numbers.

COROLLARY 6.2. *Let q be a quadratic functional on a Hilbert space $\mathfrak{H}$. Then for any x and y in $\mathfrak{H}$*

$$[q(x + y)]^{1/2} \leqq [q(x)]^{1/2} + [q(y)]^{1/2}.$$

Proof. Let L be defined as in Lemma 6.1. The inequality to be proved is obvious if either $x \notin L$ or $y \notin L$. If x and y are both in L, the Schwarz inequality holds, and the inequality to be proved follows from it the same way that the triangle inequality usually follows from the Schwarz inequality.

DEFINITION. Let $\mathfrak{A}$ be a W^*-algebra on a Hilbert space $\mathfrak{H}$, and let q be a quadratic functional on $\mathcal{H}$. We shall say q is *affiliated* with $\mathfrak{A}$ and write $q\eta\mathfrak{A}$ if $q(Ux)=q(x)$ for every $x\in\mathcal{H}$ and every unitary operator U which commutes with $\mathfrak{A}$. We shall say that q is *essentially measurable* relative to $\mathfrak{A}$ if $q\eta\mathfrak{A}$ and there exists a measurable operator T such that $q(x)=(Tx, x)$ for all x in a strongly dense subset of $\mathcal{H}$ [10, Definition 2.1]. We note that this T, if it exists, is uniquely determined by q. We shall use the notation: $T=\Theta q$.

LEMMA 6.3. *Let $\mathfrak{A}$ be a W^*-algebra on a Hilbert space $\mathcal{H}$. Let q and Q be quadratic functionals on $\mathcal{H}$ such that $q(x)\leqq Q(x)$ for all x in a strongly dense subset D_1 of $\mathcal{H}$ [10, Definition 2.1]. If Q is essentially measurable and $q\eta\mathfrak{A}$, then q is also essentially measurable.*

Proof. Let $Q(x)=(\Theta_Q x, x)$ for all x in $D\subset D_1$ where D is a strongly dense subset of $\mathcal{H}$ defined by the sequence of subspaces $\{K_n\}$ [10, Definition 2.1]. Then Q restricted to K_n is a bounded quadratic functional since Θ_Q restricted to K_n is a bounded operator by the closed graph theorem [14, p. 165, Theorem 8]. Let b be the result of polarizing q. Then for $x\in K_n$

$$b(x, x) = q(x) \leqq Q(x).$$

Thus b restricted to K_n is a bounded bilinear functional; therefore,

$$b(x, y) = (T_n x, y) \qquad \text{for } x \text{ and } y \text{ in } K_n$$

where T_n is a bounded linear operator on K_n. Furthermore, $T_1\subset T_2\subset T_3\subset \cdots$. Let T be their common extension with domain D' equal to the set-theoretic union of the K_n. Since $q\eta\mathfrak{A}$, it is clear that each $T_n\eta\mathfrak{A}$, and hence that $T\eta\mathfrak{A}$. Furthermore, T has a single-valued closure since it is a symmetric operator. Since T is bounded on K_n, it follows that T is essentially measurable. Finally $(Tx, x)=q(x)$ for $x\in D'$, and D' is a strongly dense subset of $\mathcal{H}$.

DEFINITION. Let $(\mathcal{H}, \mathfrak{A}, m)$ be a gage space and let E_0 be the maximal null projection. If q is a quadratic functional affiliated with $\mathfrak{A}$, we define:

$$m(q) = \begin{cases} m(\Theta_{q\circ E_0}) & \text{if } q\circ E_0 \text{ is essentially measurable,} \\ +\infty & \text{otherwise.} \end{cases}$$

COROLLARY 6.4. *Let $(\mathcal{H}, \mathfrak{A}, m)$ be a gage space and let q and Q be two quadratic functionals affiliated with $\mathfrak{A}$. If $q(x)\leqq Q(x)$ for all $x\in\mathcal{H}$, then $m(q)\leqq m(Q)$.*

Proof. Immediate from Lemma 6.3.

LEMMA 6.5. *Let S and T be non-negative self-adjoint operators on a Hilbert space $\mathcal{H}$. Let $\mathfrak{D}$ be a dense linear subset of $\mathcal{H}$ contained in the domain of $T^{1/2}$ such that $T^{1/2}$ is the closure of its restriction to $\mathfrak{D}$. If $q_S(x)\leqq q_T(x)$ for all $x\in\mathfrak{D}$, then $q_S(x)\leqq q_T(x)$ for all $x\in\mathcal{H}$.*

Proof. By hypothesis $\|S^{1/2}x\|\leqq\|T^{1/2}x\|$ for all $x\in\mathfrak{D}$. It follows that there exists a bounded linear operator W with $\|W\|\leqq 1$ such that $S^{1/2}x=WT^{1/2}x$ for all $x\in\mathfrak{D}$. If $x\notin$domain of $T^{1/2}$, automatically $q_S(x)\leqq q_T(x)$. Suppose

$x \in$ domain of $T^{1/2}$. By hypothesis there exists a sequence $\{x_n\}$ of elements of $\mathfrak{D}$ such that $x_n \to x$ and $T^{1/2}x_n \to T^{1/2}x$. But then $S^{1/2}x_n = WT^{1/2}x_n \to WT^{1/2}x$. Since $S^{1/2}$ is closed, it follows that x is in the domain of $S^{1/2}$ and $S^{1/2}x = WT^{1/2}x$. Hence

$$q_S(x) = \|S^{1/2}x\|^2 \leqq \|T^{1/2}x\|^2 = q_T(x).$$

Corollary 6.6. *Let $\mathfrak{A}$ be a W^*-algebra on a Hilbert space $\mathfrak{H}$, and let S and T be non-negative self-adjoint operators affiliated with $\mathfrak{A}$. Suppose T is measurable. If $q_S(x) \leqq q_T(x)$ for all x in a strongly dense subset of $\mathfrak{H}$, then*

$$q_S(x) \leqq q_T(x) \qquad \text{for all } x \in \mathfrak{H}.$$

The collection of all essentially measurable quadratic functionals is too wide a class with which to formulate the monotone convergence theorem. In the next definition we direct attention to a special sort of quadratic functional. We shall say that one quadratic functional q is $\leqq$ another Q when $q(x) \leqq Q(x)$ for all $x \in \mathfrak{H}$. We note that the pointwise least upper bound of a directed set of quadratic functionals (directed by $\leqq$) is again a quadratic functional. Furthermore, if all the quadratic functionals in the directed set are affiliated with a W^*-algebra, so is the least upper bound.

Definition. We shall say that a quadratic functional is *restrained* if it is the least upper bound of a collection of standard functionals corresponding to bounded operators.

Lemma 6.7. *The standard quadratic functional corresponding to any non-negative self-adjoint operator is restrained.*

Proof. Let T be a non-negative self-adjoint operator. We may assume without loss of generality that T is multiplication by a non-negative measurable function ϕ operating on L_2 of a measure space. It is easy to see that $q_T(f) = \int \phi |f|^2$ for all f in L_2. Let

$$\phi_n(p) = \begin{cases} \phi(p) & \text{if } \phi(p) \leqq n, \\ 0 & \text{otherwise,} \end{cases}$$

and let T_n be the corresponding multiplication operator. Then, by the monotone convergence theorem of ordinary measure theory

$$q_{T_n}(f) = \int \phi_n |f|^2 \nearrow \int \phi |f|^2 = q_T(f).$$

Lemma 6.8. *Let T be a non-negative self-adjoint operator on a Hilbert space $\mathfrak{H}$. Let $\mathfrak{D}$ be a dense linear subset of $\mathfrak{H}$ contained in the domain of $T^{1/2}$ such that $T^{1/2}$ is the closure of its restriction to $\mathfrak{D}$. If q is a restrained quadratic functional such that $q(x) \leqq q_T(x)$ for all $x \in \mathfrak{D}$, then*

$$q(x) \leqq q_T(x) \qquad \textit{for all } x \in \mathfrak{H}.$$

Proof. This lemma is an immediate consequence of Lemma 6.5 and the definition of restrained quadratic functional.

COROLLARY 6.9. *Let $\mathfrak{A}$ be a W^*-algebra on a Hilbert space $\mathfrak{H}$, and let T be a non-negative measurable operator. If q is a restrained quadratic functional such that $q(x) \leqq q_T(x)$ for all x in a strongly dense subset of $\mathfrak{H}$, then*

$$q(x) \leqq q_T(x) \qquad \textit{for all } x \in \mathfrak{H}.$$

THEOREM 6.10. *Let $(\mathfrak{H}, \mathfrak{A}, m)$ be a gage space. Let $\{q_\alpha\}$ be a monotone non-decreasing directed system of quadratic functionals affiliated with $\mathfrak{A}$. If the q_α are all restrained, then*

$$m\left(\sup_\alpha q_\alpha\right) = \sup_\alpha m(q_\alpha).$$

Proof. The treatment of the maximal null projection E_0 in the definition of m applied to a quadratic functional is such that there is no loss of generality in assuming that m is a nonsingular gage (i.e., $E_0=0$). Corollary 6.4 implies that

$$m\left(\sup_\alpha q_\alpha\right) \geqq \sup_\alpha m(q_\alpha).$$

If $\sup_\alpha m(q_\alpha) = +\infty$, there is nothing further to prove. So we assume that $\sup_\alpha m(q_\alpha) = M < +\infty$. In that event, each q_α is essentially measurable and $\{\Theta_{q_\alpha}\}$ is Cauchy in $L_1(\mathfrak{H}, \mathfrak{A}, m)$. Hence, by [10, Theorem 13], $\{\Theta_{q_\alpha}\}$ converges in $L_1(\mathfrak{H}, \mathfrak{A}, m)$ to some operator T in $L_1(\mathfrak{H}, \mathfrak{A}, m)$. Given any index $\alpha(0)$, there exists a sequence $\alpha(0) \leqq \alpha(1) \leqq \alpha(2) \leqq \cdots$ such that $\Theta_{q_{\alpha(n)}} \to T$ in $L_1(\mathfrak{H}, \mathfrak{A}, m)$. But by [10, Corollary 13.1], there exists a subsequence converging to T n.e. It follows then from [10, Corollary 13.2], that $T \geqq \Theta_{q_{\alpha(0)}}$. Then by Corollary 6.9 $q_T \geqq q_{\alpha(0)}$. But since this is so for all $\alpha(0)$, $q_T \geqq \sup_\alpha q_\alpha$. Hence by Corollary 6.4,

$$M = m(q_T) \geqq m\left(\sup_\alpha q_\alpha\right).$$

III. PRODUCT GAGES

7. A noncommutative Fubini theorem. Throughout this section we shall be considering two gage spaces $\Gamma_1 = (\mathfrak{H}_1, \mathfrak{A}_1, m_1)$ and $\Gamma_2 = (\mathfrak{H}_2, \mathfrak{A}_2, m_2)$. On the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ we shall consider the W^*-algebra $\mathfrak{A}_1 \otimes \mathfrak{A}_2$. In [10, §5.7], Segal defines the product gage when Γ_1 and Γ_2 are standard gage spaces, and then proves a version of the Fubini theorem [10, Corollary 22.1]. We wish to define the product gage for arbitrary gage spaces, and to prove a Fubini theorem in which the marginal integral is explicitly identified.

DEFINITION. Let $\mathcal{E}$ be the algebra of all finite sums $\sum_{k=1}^n A_k \otimes B_k$ where A_k is an elop [10, Definition 3.1] in $\mathfrak{A}_1$ and B_k is an elop in $\mathfrak{A}_2$. By the *product* *gage* of m_1 and m_2, we shall mean the gage $m=m_1\times m_2$ on $\mathfrak{A}_1\otimes\mathfrak{A}_2$ such that

$$\mathcal{E}\subset L_1(\mathfrak{H}_1\otimes\mathfrak{H}_2,\ \mathfrak{A}_1\otimes\mathfrak{A}_2,\ m)$$

and $m(\sum_{k=1}^n A_k\otimes B_k)=\sum_{k=1}^n m_1(A_k)m_2(B_k)$ for any elops A_k in $\mathfrak{A}_1$ and any elops B_k in $\mathfrak{A}_2$. The uniqueness of m follows from Corollary 1.3. The existence of m will be a consequence of Theorem 7.3.

Let T be a non-negative self-adjoint operator affiliated with $\mathfrak{A}_1\otimes\mathfrak{A}_2$. Consider, in the notation of §6, $q_T(x\otimes y)$. This expression is a quadratic functional in either variable when the other is fixed. Furthermore, for fixed x, $q_T(x\otimes\cdot)$ is affiliated with $\mathfrak{A}_2$. We define a quadratic functional $\Xi_2(T)$ on $\mathfrak{H}_1$ by setting

$$[\Xi_2(T)](x)=m_2(q_T(x\otimes\ \cdot)).$$

Then $\Xi_2(T)\eta\mathfrak{A}_1$. If $q_T(x\otimes\cdot)$ is essentially measurable with respect to $\mathfrak{A}_2$, we shall write

$$^xT=\Theta_{q_T(x\otimes\cdot)}.$$

If $\Xi_2(T)$ is essentially measurable with respect to $\mathfrak{A}_1$, we write $\xi_2(T)=\Theta_{\Xi_2(T)}$. If S is a measurable operator with respect to $\mathfrak{A}_1\otimes\mathfrak{A}_2$ which is a linear combination of non-negative measurable operators T_j with $q_{T_j(x\otimes\cdot)}$ essentially measurable, we define ^xS to be the corresponding linear combination of the ^xT_j. If the T_j can be so chosen that the $\Xi_2(T_j)$ are essentially measurable, we define $\xi_2(S)$ to be the corresponding linear combination of the $\xi_2(T_j)$. That these two notions are well-defined comes from the fact that

$$q_{\lambda T+\mu T'}(x\otimes\ \cdot)=\lambda q_T(x\otimes\ \cdot)+\mu q_{T'}(x\ \otimes\ \cdot)$$

and $\Xi_2(\lambda T+\mu T')=\lambda\Xi_2(T)+\mu\Xi_2(T')$ where λ and μ are non-negative numbers and T and T' are non-negative self-adjoint operators affiliated with $\mathfrak{A}_1\otimes\mathfrak{A}_2$. If $\xi_2(S)$ exists, then ^xS exists and $(\xi_2(S)x,x)=m_2(^xS)$ for a strongly dense set of x. If S is bounded, then ^xS exists and is bounded for all x.

LEMMA 7.1. *If T is a non-negative self-adjoint operator affiliated with $\mathfrak{A}_1\otimes\mathfrak{A}_2$ and $x\in\mathfrak{H}_1$, then $q_T(x\otimes\cdot)$ is a restrained quadratic functional on $\mathfrak{H}_2$.*

Proof. Let $T=\int_0^\infty\lambda dE_\lambda$ be the spectral resolution of T. Set $T_n=\int_0^n\lambda dE_\lambda$. Then

$$q_T(x\otimes y)=\int_0^\infty\lambda d(E_\lambda(x\otimes y),\ x\otimes y)$$

$$=\sup_n\int_0^n\lambda d(E_\lambda(x\otimes y),\ x\otimes y)=\sup_n q_{T_n}(x\otimes y).$$

But $q_{T_n}(x\otimes y)\leqq n\|x\|^2\|y\|^2$. Hence $q_T(x\otimes\cdot)$ is the least upper bound of bounded quadratic functionals $q_{T_n}(x\otimes\cdot)$.

LEMMA 7.2. *If T is a non-negative self-adjoint operator affiliated with $\mathfrak{A}_1 \otimes \mathfrak{A}_2$, then $\Xi_2(T)$ is a restrained quadratic functional on $\mathfrak{H}_1$.*

Proof. Defining T_n as in the proof of Lemma 7.1, we have that $q_T(x\otimes\cdot)$ is the least upper bound of $q_{T_n}(x\otimes\cdot)$. Then by definition of Ξ_2 and Theorem 6.10, we see that $\Xi_2(T)$ is the least upper bound of $\Xi_2(T_n)$. Therefore, it is sufficient to prove the lemma when T is bounded.

Let $\mathfrak{F}_2$ be the family of all projections Q in $\mathfrak{A}_2$ such that $m_2(Q) < +\infty$. Now if $x \in \mathfrak{H}_1$,

$$^x[(I \otimes Q)T(I \otimes Q)] = Q(^xT)Q,$$

and so

$$[\Xi_2((I \otimes Q)T(I \otimes Q))](x) = m_2(Q(^xT)Q).$$

But $\sup_{Q\in\mathfrak{F}_2} m_2(Q(^xT)Q) = m_2(^xT)$ by [10, Lemma 13.1]. It follows that $\Xi_2(T)$ is the least upper bound of $\Xi_2((I\otimes Q)T(I\otimes Q))$ as Q runs over $\mathfrak{F}_2$. However

$$m_2(Q(^xT)Q) \leqq \|^xT\|_\infty m_2(Q) \leqq \|T\|_\infty m_2(Q)\|x\|^2.$$

Hence $\Xi_2((I\otimes Q)T(I\otimes Q))$ is a bounded quadratic functional for $Q \in \mathfrak{F}_2$.

THEOREM 7.3. *Let $(\mathfrak{H}_j, \mathfrak{A}_j, m_j)(j=1, 2)$ be gage spaces, and let $\mathfrak{F}_j$ be the family of all m_j-finite projections in $\mathfrak{A}_j$. Let $\mathfrak{G}$ be the algebra of all operators of the form $(P\otimes Q)T(P\otimes Q)$ where T runs over $\mathfrak{A}_1\otimes\mathfrak{A}_2$, P runs over $\mathfrak{F}_1$, and Q runs over $\mathfrak{F}_2$. Then for all $S \in \mathfrak{G}$, $\xi_2(S)$ exists and is in $L_1(\mathfrak{H}_1, \mathfrak{A}_1, m_1)$. Furthermore, there exists a gage m on $\mathfrak{A}_1\otimes\mathfrak{A}_2$ such that $\mathfrak{G}\subset L_1(\mathfrak{H}_1\otimes\mathfrak{H}_2, \mathfrak{A}_1\otimes\mathfrak{A}_2, m)$ and $m(S) = m_1(\xi_2(S))$ for $S\in\mathfrak{G}$.*

Proof. We show first that when $S\in\mathfrak{G}$, $\xi_2(S)$ exists and is in $L_1(\mathfrak{H}_1, \mathfrak{A}_1, m_1)$. It is sufficient to show this when $S \geqq 0$. Choose projections $P\in\mathfrak{F}_1$ and $Q\in\mathfrak{F}_2$ such that S lives on $P\otimes Q$. Then $q_S(x\otimes y) \leqq \|S\|_\infty q_{P\otimes Q}(x\otimes y)$ and so $\Xi_2(S) \leqq \|S\|_\infty \Xi_2(P\otimes Q)$. But $\Xi_2(P\otimes Q)$ is a bounded quadratic functional and $\xi_2(P\otimes Q) = m_2(Q)P$. Hence $\Xi_2(S)$ is bounded and $\xi_2(S) \leqq \|S\|_\infty m_2(Q)P$. Therefore $\xi_2(S) \in L_1(\mathfrak{H}_1, \mathfrak{A}_1, m_1)$.

Set $l(S) = m_1(\xi_2(S))$ for $s\in\mathfrak{G}$. We are going to apply Theorem 1.1 to l. Hypothesis (2) is obviously satisfied if we take $E_\nu = P\otimes Q$ where $\nu = (P, Q)$ runs over the directed set $\mathfrak{F}_1\times\mathfrak{F}_2$.

As for hypothesis (1), it is sufficient to show that for $P\in\mathfrak{F}_1$ and $Q\in\mathfrak{F}_2$ the map $T\to l((P\otimes Q)T(P\otimes Q))$ is weakly continuous when restricted to the unit ball of $\mathfrak{G}$. In the rest of this paragraph we make the convention that T runs over the unit ball of $\mathfrak{G}$. Clearly ^xT is weakly continuous in T and $\|^xT\|_\infty \leqq \|x\|^2$. Therefore [10, Theorem 14] and [3, Theorem 2] $m_2(^x[(I\otimes Q)T(I\otimes Q)]) = m_2(Q(^xT))$ is weakly continuous in T. Furthermore, $|m_2(Q(^xT))| \leqq m_2(Q) \cdot \|x\|^2$; so $\xi_2((I\otimes Q)T(I\otimes Q))$ is a bounded operator and

$$(\xi_2((I \otimes Q)T(I \otimes Q))x, x) = m_2(Q(^xT)) \qquad \text{for all } x \in \mathfrak{H}_1.$$

It follows that $\xi_2((I\otimes Q)T(I\otimes Q))$ is weakly continuous in T and

$$\|\xi_2((I \otimes Q)T(I \otimes Q))\|_\infty \leqq 2m(Q).$$

Hence [10, Theorem 14] and [3, Theorem 2]

$$\begin{aligned} l((P \otimes Q)T(P \otimes Q)) &= m_1(\xi_2((P \otimes Q)T(P \otimes Q))) \\ &= m_1(P\xi_2((I \otimes Q)T(I \otimes Q))) \text{ is weakly continuous in } T. \end{aligned}$$

It remains only to show that l is a central linear functional. It is sufficient to show that for $P\in\mathfrak{F}_1$ and $Q\in\mathfrak{F}_2$

$$(*) \qquad l((P \otimes Q)ST(P \otimes Q)) = l((P \otimes Q)TS(P \otimes Q))$$

for all S and T in $\mathcal{G}$. Now this is obvious if S and T are in $\mathcal{E}$. But $\mathcal{E}$ is weakly dense in $\mathfrak{A}_1\otimes\mathfrak{A}_2$, and so by [6] the unit ball of $\mathcal{E}$ is weakly dense in the unit ball of $\mathfrak{A}_1\otimes\mathfrak{A}_2$. Using this, equation (*) follows by two applications of the continuity property of $T\rightarrow l((P\otimes Q)T(P\otimes Q))$ proved in the last paragraph.

The present theorem now follows from Theorem 1.1.

We are now in a position to prove the main result of this section—a non-commutative Fubini theorem.

THEOREM 7.4. *Let $\Gamma_1=(\mathcal{H}_1, \mathfrak{A}_1, m_1)$ and $\Gamma_2=(\mathcal{H}_2, \mathfrak{A}_2, m_2)$ be gage spaces. Let $\Gamma=(\mathcal{H}_1\otimes\mathcal{H}_2, \mathfrak{A}_1\otimes\mathfrak{A}_2, m)$ be the product gage space. If T is any non-negative self-adjoint operator affiliated with $\mathfrak{A}_1\otimes\mathfrak{A}_2$, then $m(T)=m_1(\Xi_2(T))$. If $T\in L_1(\Gamma)$, then $\xi_2(T)$ exists and lies in $L_1(\Gamma_1)$ and $m(T)=m_1(\xi_2(T))$.*

Proof. The last sentence is clearly a consequence of the sentence before, so we consider a non-negative self-adjoint operator T affiliated with $\mathfrak{A}_1\otimes\mathfrak{A}_2$.

CASE I. *T is bounded.* Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be as in Theorem 7.3. In the proof of this case we restrict P to run over $\mathfrak{F}_1$ and Q to run over $\mathfrak{F}_2$. By Theorem 7.3

$$m((P \otimes Q)T(P \otimes Q)) = m_1(\xi_2((P \otimes Q)T(P \otimes Q)))$$

and by [10, Lemma 13.1],

$$m(T) = \sup_{P,Q} m((P \otimes Q)T(P \otimes Q)).$$

But

$$\begin{aligned} &\sup_{P,Q} m_1(\xi_2((P \otimes Q)T(P \otimes Q))) \\ &\qquad = \sup_Q \sup_P m_1(P\xi_2((I \otimes Q)T(I \otimes Q))) \\ &\qquad = \sup_Q m_1(\xi_2((I \otimes Q)T(I \otimes Q))) \qquad \text{by [10, Lemma 13.1]} \\ &\qquad = \sup_Q m_1(\Xi_2((I \otimes Q)T(I \otimes Q))) \\ &\qquad = m_1(\Xi_2(T)) \qquad \text{by Theorem 6.10 and Lemma 72.} \end{aligned}$$

since we saw in the proof of Lemma 7.2 that $\Xi_2(T)$ is the least upper bound of $\Xi_2((I\otimes Q)T(I\otimes Q))$.

CASE II. *T may be unbounded.* Let T_n be as in the proof of Lemma 7.1. In the proof of Lemma 7.2, we saw that $\Xi_2(T)$ is the least upper bound of $\Xi_2(T_n)$. Hence we have

$$m(T) = \sup_n m(T_n) = \sup_n m_1(\Xi_2(T_n)) \qquad \text{by Case I}$$

$$= m_1(\Xi_2(T)) \qquad \text{by Theorem 6.10 and Lemma 7.2.}$$

COROLLARY 7.5. *The product gage m is nonsingular if and only if both m_1 and m_2 are nonsingular.*

Proof. If either m_1 or m_2 is singular, then m is clearly singular. Suppose m_1 and m_2 are nonsingular. Let P be a projection with $m(P)=0$. Then $\xi_2(P)=0$ and hence ${}^xP=0$ for a dense set of x. In other words $P=0$.

8. **Tensor products of unbounded operators.** We continue the notation of §7.

DEFINITION. If S and T are operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, we define their *algebraic tensor product*, denoted by $S\otimes_{\text{alg}}T$, to be the smallest linear extension of the map $x\otimes y\to Sx\otimes Ty$ where x is in the domain of S and y is in the domain of T. If S and T are closed operators, we define $S\otimes T$ to be the closure of $S\otimes_{\text{alg}}T$. This closure is single-valued by the following theorem.

THEOREM 8.1. *Let S and T be closed, densely defined operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $S\otimes T$ is single-valued and $(S\otimes T)^*=S^*\otimes T^*$.*

Proof. Suppose x is in the domain of S and y is in the domain of T. For x' in the domain of S^* and y' in the domain of T^*, we make the following calculation:

$$\begin{aligned}(S^*x'\otimes T^*y', x\otimes y) &= (S^*x', x)(T^*y', y) = (x', Sx)(y', Ty)\\ &= (x'\otimes y', Sx\otimes Ty).\end{aligned}$$

It follows from this that

$$(S\otimes_{\text{alg}} T)(x\otimes y) = (S^*\otimes_{\text{alg}} T^*)^*(x\otimes y).$$

Therefore $S\otimes_{\text{alg}}T\subset(S^*\otimes_{\text{alg}}T^*)^*$. Hence $S\otimes_{\text{alg}}T$ has a single-valued closure and $S\otimes T\subset(S^*\otimes T^*)^*$.

Suppose that S and T are self-adjoint. From what has been shown $S\otimes T$ is a closed symmetric operator. To show that $S\otimes T$ is self-adjoint it is sufficient to show that the ranges of $S\otimes T\pm i(I\otimes I)$ are dense in $\mathcal{H}_1\otimes\mathcal{H}_2$. But it is clear from the fact that $S\pm iI$ and $T\pm iI$ have dense ranges that even $S\otimes_{\text{alg}}T\pm i(I\otimes I)$ have dense ranges.

Finally let S and T be arbitrary closed densely defined operators. Let $S=U|S|$ and $T=V|T|$ be their polar decompositions. Then $S\otimes T=(U\otimes V)(|S|\otimes|T|)$ is the polar decomposition of $S\otimes T$. It is easy to see that the following calculation is valid:

$$(S \otimes T)^* = (|S| \otimes |T|)(U^* \otimes V^*) = (|S|U^*) \otimes (|T|V^*) = S^* \otimes T^*.$$

THEOREM 8.2. *Let S and T be self-adjoint operators on $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. If the spectral resolutions of S and T are $S=\int_{-\infty}^{+\infty}\lambda dE_\lambda$ and $T=\int_{-\infty}^{+\infty}\mu dF_\mu$, then*

$$S \otimes T = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda\mu d(E_\lambda \otimes F_\mu).$$

Proof. Let $A=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\lambda\mu d(E_\lambda\otimes F_\mu)$. Suppose x is in the domain of S and y is in the domain of T. Then

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda^2\mu^2 d((E_\lambda \otimes F_\mu)(x \otimes y), x \otimes y)$$
$$= \int_{-\infty}^{+\infty} \lambda^2 d(E_\lambda x, x) \int_{-\infty}^{+\infty} \mu^2 d(F_\mu y, y) < +\infty.$$

Therefore, $x\otimes y$ is in the domain of A. For any $x'\in\mathfrak{H}_1$ and $y'\in\mathfrak{H}_2$,

$$\begin{aligned}((S \otimes T)(x \otimes y), x' \otimes y') &= \int_{-\infty}^{+\infty} \lambda d(E_\lambda x, x') \int_{-\infty}^{+\infty} \mu d(F_\mu y, y') \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda\mu d((E_\lambda \otimes F_\mu)(x \otimes y), x' \otimes y') \\ &= A(x \otimes y), (x' \otimes y').\end{aligned}$$

Hence $(S\otimes T)(x\otimes y)=A(x\otimes y)$. From this it follows that $S\otimes T\subset A$, and therefore, $S\otimes T=A$ since both are self-adjoint.

COROLLARY 8.3. *Suppose m_1 and m_2 are nonsingular gages. If $S\in L_1(\Gamma_1)$ and $T\in L_1(\Gamma_2)$, then $S\otimes T\in L_1(\Gamma)$ and $\|S\otimes T\|_1=\|S\|_1\|T\|_1$. If $S\in L_2(\Gamma_1)$ and $T\in L_2(\Gamma_2)$, then $S\otimes T\in L_2(\Gamma)$ and $\|S\otimes T\|_2=\|S\|_2\|T\|_2$.*

Proof. Let $|S|=\int_0^\infty\lambda dE_\lambda$ and $|T|=\int_0^\infty\mu dF_\mu$ be the spectral resolutions of $|S|$ and $|T|$. We have seen in the proof of Theorem 8.1 that $|S\otimes T| = |S|\otimes|T|$. Hence for $p=1, 2$

$$\|S \otimes T\|_p^p = \int_0^\infty\int_0^\infty (\lambda\mu)^p\, dm(E_\lambda \otimes F_\mu)$$

by Theorem 8.2 and [10 Corollary 12.6],

$$= \int_0^\infty \lambda^p\, dm_1(E_\lambda) \int_0^\infty \mu^p dm_2(F_\mu)$$

$$= \|S\|_p^p\|T\|_p^p.$$

When $S \in L_1(\Gamma_1)$ and $T \in L_1(\Gamma_2)$ and $A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$, then the following formal calculation is invalid since $I \otimes T$ may not be measurable:

$$\begin{aligned}(\xi_2(A(S \otimes T))x, x) &= (\xi_2(A(I \otimes T)Sx, x) \\ &= m_2({}^{Sx,x}[A(I \otimes T)]) = m_2(({}^{Sx,x}A)T)\end{aligned}$$

where we have used the notation ${}^{x_1,x_2}B$ for the result of polarizing ${}^{x}B$ in the variable x. Nevertheless, as the following theorem shows, the result is essentially correct. It will be of use to us in §9.

THEOREM 8.4. *Suppose m_1 and m_2 are nonsingular gages. If $S \in L_1(\Gamma_1)$ and $T \in L_1(\Gamma_2)$ and $A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2$, then $\xi_2(A(S \otimes T)) = RS$ where R is the bounded operator on $\mathfrak{H}_1$ such that $(Rx, x) = m_2({}^{x}AT)$ for $x \in \mathfrak{H}_1$.*

Proof. Since $A(S \otimes T) \in L_1(\Gamma)$, $\xi_2(A(S \otimes T))$ exists. Therefore $(\xi_2(A(S \otimes T))x, x) = m_2({}^{x}[A(S \otimes T)])$ for a strongly dense set of vectors x in $\mathfrak{H}_1$. By polarizing this equation in x, we get that for a strongly dense set of vectors x_1 and x_2 in $\mathfrak{H}_1$,

$$\xi_2(A(S \otimes T)x_1, x_2) = m_2({}^{x_1,x_2}[A(S \otimes T)])$$

where for a strongly dense set of vectors y_1 and y_2 in $\mathfrak{H}_2$,

$$({}^{x_1,x_2}[A(S \otimes T)]y_1, y_2) = (A(Sx_1 \otimes Ty_1), x_2 \otimes y_2) = (({}^{Sx_1,x_2}A)Ty_1, y_2).$$

Hence ${}^{x_1,x_2}[A(S \otimes T)] = ({}^{Sx_1,x_2}A)T$. It follows that

$$(\xi_2(A(S \otimes T))x_1, x_2) = m(({}^{Sx_1,x_2}A)T) = (RSx_1, x_2).$$

Therefore, $\xi_2(A(S \otimes T)) = RS$.

IV. HARMONIC ANALYSIS AND DUALITY FOR UNIMODULAR GROUPS

9. **Harmonic analysis.** Throughout the following two sections $\mathcal{G}$ will stand for a locally compact unimodular group; $\mathfrak{H}$ will stand for the Hilbert space $L_2(\mathcal{G})$ formed with respect to Haar measure on $\mathcal{G}$; L_a where $a \in \mathcal{G}$ will stand for left translation by a acting on $\mathfrak{H}$:

$$(L_af)(x) = f(a^{-1}x);$$

and $\mathfrak{L}$ will stand for the W^*-algebra generated by all the L_a with a in $\mathcal{G}$. If $f \in L_1(\mathcal{G})$, we shall write L_f for the operator of left convolution by f acting on $\mathfrak{H}$:

$$(L_fg)(x) = (f * g)(x) = \int f(y)g(y^{-1}x)dy.$$

The mapping $f \to L_f$ is to be thought of as a "global" Fourier transform. A greater formal analogy with the abelian case is manifest in the formula

$$L_f = \int f(x)L_xdx, \qquad f \in L_1(\mathcal{G})$$

where the integral is interpreted in the weak sense. The operators L_f are in $\mathfrak{L}$, and $\mathfrak{L}$ is the W^*-algebra they generate. The latter is easily seen by letting ψ_N be the characteristic function of a measurable subset N of $\mathcal{G}$ divided by the measure of N; then as N runs over the directed set of compact neighborhoods of a, $L_{\psi_N} \rightarrow L_a$ weakly.

LEMMA 9.1. *If f and g are in $L_1(\mathcal{G})$ and $L_f = L_g$, then $f = g$ a.e.*

Proof. Let $h \in L_2(\mathcal{G}) \cap L_\infty(\mathcal{G})$. Then $f * h = g * h$ *a.e.* But both sides of this equation are continuous, and so their values at e, the identity of $\mathcal{G}$, are equal:

$$\int f(y)h(y^{-1})dy = \int g(y)h(y^{-1})dy.$$

Therefore $f = g$ a.e.

In [9], Segal introduced a gage on the W^*-algebra $\mathfrak{L}$. The existence of this gage is an easy consequence of Theorem 1.1.

THEOREM 9.2. *There exists a unique gage m on $\mathfrak{L}$ such that whenever f is a continuous positive definite function in $L_1(\mathcal{G})$, then $L_f \in L_1(\mathcal{H}, \mathfrak{L}, m)$ and $m(L_f) = f(e)$, where e is the identity of $\mathcal{G}$.*

Proof. Let $\mathcal{P}$ be the set of all linear combinations of continuous positive definite functions in $L_1(\mathcal{G})$, and let $\mathcal{A}$ be the set of all L_f with f in $\mathcal{P}$. Since $\mathcal{P}$ contains all convolutions of two functions in $L_1(\mathcal{G}) \cap L_2(\mathcal{G})$, the set $\mathcal{A}$ is a subalgebra of $\mathfrak{L}$. It is clearly a self-adjoint algebra. Furthermore, $\mathcal{A}$ is weakly dense in $\mathfrak{L}$; for, taking ψ_N as above, it is easy to see that

$$L_{\Psi_N * \Psi_{aN}} \rightarrow L_a \qquad \text{weakly}$$

as N runs over the compact neighborhoods of e.

We define l on $\mathcal{A}$ by the formula $l(L_f) = f(e)$; by Lemma 9.1, l is a single-valued. Then l is non-negative since

$$l(L_f^* L_f) = (f^* * f)(e) = \|f\|_2^2 \geqq 0$$

where $f^*(x) = [f(x^{-1})]^-$. Furthermore, l is central since

$$l(L_g L_f) = (g * f)(e) = \int g(y)f(y^{-1})dy$$

$$= \int f(y)g(y^{-1})dy = l(L_f L_g).$$

As for hypothesis (1) of Theorem 1.1,

$$l(L_g^* L_h L_f) = \int (L_h f)(y)[g^*(y^{-1})]^- dy = (L_h f, g)$$

which is clearly weakly continuous in L_h. To satisfy hypothesis (2) of Theorem 1.1, we set $E_N = L\psi_N * \psi_N$ for N running over the directed set of compact neighborhoods of e. Then

$$l(L_f E_N) = \int f(y)(\psi_N * \psi_N)(y^{-1})dy$$

and

$$l(\overset{*}{E}_N L_f E_N) = \int f(y)(\psi_N * \psi_N * \psi_N * \psi_N)(y^{-1})dy,$$

both of which are easily seen to converge to $f(e) = l(L_f)$.

Hence by Theorem 1.1 there exists a gage m on $\mathcal{L}$ such that $\mathfrak{A} \subset L_1(\mathfrak{H}, \mathcal{L}, m)$ and $m(L_f) = f(e)$ for $f \in \mathcal{P}$. The uniqueness of m follows from Corollary 1.3.

Definition. The gage m on $\mathcal{L}$ described in Theorem 9.2 will be called the *dual gage*. We shall call the gage space $\Gamma = (\mathfrak{H}, \mathcal{L}, m)$ the *dual gage space* to $\mathcal{G}$.

Corollary 9.3. *If $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$, then $L_f \in L_2(\Gamma)$ and $\|L_f\|_2 = \|f\|_2$. The set of all such L_f is dense in $L_2(\Gamma)$.*

Proof. If $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$, then $f^* * f \in \mathcal{P}$. Hence $\|f\|_2^2 = (f^* * f)(e) = m(L_f^* L_f) = \|L_f\|_2^2$. To show that the set $\mathfrak{M}$ of all such L_f is dense in $L_2(\Gamma)$, we let $T \in L_2(\Gamma)$ be $\perp \mathfrak{M}$, and apply the following remark. Note that $\mathfrak{M}$ is weakly dense in $\mathcal{L}$ by an argument in the proof of Theorem 9.2.

Remark 9.4. In this remark we interrupt the notation of this section. Suppose that $(\mathfrak{H}, \mathfrak{A}, m)$ is any nonsingular gage space and that $\mathfrak{M}$ is a strongly dense *-subalgebra of $\mathfrak{A}$ which is contained in $L_2(\mathfrak{H}, \mathfrak{A}, m)$. If an operator $S \in L_1(\mathfrak{H}, \mathfrak{A}, m)$ and an operator $T \in L_2(\mathfrak{H}, \mathfrak{A}, m)$ are such that $m(AS) = m(AT)$ for every $A \in \mathfrak{M}$, then $S = T$. For if $B \in \mathfrak{M}$, then BS and BT are integrable. The continuity property of the indefinite integral [10, Theorem 14] together with the result in [6], allow us to conclude, from the fact that $m(ABS) = m(ABT)$ for all $A \in \mathfrak{M}$, that $m(ABS) = m(ABT)$ for all $A \in \mathfrak{A}$. In other words $\|BS - BT\|_1 = 0$, and so $BS = BT$, for all $B \in \mathfrak{M}$. Since $\mathfrak{M}$ is strongly dense in $\mathfrak{A}$, letting $B \to I$, we see that S coincides with T on the intersection of their domains, and therefore $S = T$.

Resuming the notation of this section, we now define L_f when $f \in L_2(\mathcal{G})$.

Definition. We use the notation $f \to L_f$ for the unitary map of $L_2(\mathcal{G})$ onto $L_2(\Gamma)$ which extends the map $f \to L_f$ for $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$.

Lemma 9.5. *If $f \in L_2(\mathcal{G})$ and $T \in \mathcal{L}$, then $TL_f = L_{Tf}$. (Cf. [10, Corollary 16.3]).*

Proof. First we show that m is a nonsingular gage. Suppose P is a projection in $\mathcal{L}$ such that $m(P) = 0$. We saw in the proof of Theorem 9.2

that $\{L_f: f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})\}$ is weakly dense in $\mathfrak{L}$. By [6], we can choose a directed system $\{h_j\}$ of elements of $L_1(\mathcal{G}) \cap L_2(\mathcal{G})$ such that $L_{h_j} \to P$ strongly and $\|L_{h_j}\|_\infty \leqq 1$. If f and g are in $L_1(\mathcal{G}) \cap L_2(\mathcal{G})$, then

$$\begin{aligned} (Pf, g) &= \lim_j (L_{h_j}f, g) = \lim_j (h_j, g * f^*) \\ &= \lim_j m([L_{g*f^*}]^* L_{h_j}) && \text{by Corollary 9.3} \\ &= m((L_{g*f^*})^* P) && \text{by [10, Theorem 14]} \\ &= 0. \end{aligned}$$

Hence $P=0$.

In proving this lemma we suppose first that $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$. Choose a directed system $\{h_j\}$ of elements of $L_1(\mathcal{G}) \cap L_2(\mathcal{G})$ such that $L_{h_j} \to T$ weakly and $\|L_{h_j}\|_\infty \leqq \|T\|_\infty$. It is elementary that

$$L_{h_j} L_f = L_{L_{h_j} f}.$$

But $L_{h_j} f \to Tf$ and hence by Corollary 9.3,

$$L_{L_{h_j} f} \to L_{Tf} \text{ in } L_2(\Gamma).$$

On the other hand, $L_{h_j} L_f \to TL_f$ weakly in $L_2(\Gamma)$, for if $S \in L_2(\Gamma)$ $m(L_{h_j} L_f S^*) \to m(TL_f S^*)$ by [10, Theorem 14]. It follows that $TL_f = L_{Tf}$ when $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$. But $f \to TL_f$ and $f \to L_{Tf}$ are both continuous mappings from $L_2(\mathcal{G})$ into $L_2(\Gamma)$. Hence $TL_f = L_{Tf}$ for all $f \in L_2(\mathcal{G})$.

Just as in the abelian case, there is here a natural back transform formula. If $F \in L_1(\Gamma)$, then its back transform is the function f on $\mathcal{G}$ given by

$$f(x) = m(L_x^* F).$$

The function f is clearly a continuous function on $\mathcal{G}$, bounded by $\|F\|_1$. The following theorem is the analogue of the Parseval formula.

THEOREM 9.6. *Let $f \in L_1(\mathcal{G})$ and $G \in L_1(\Gamma)$. Set $F = L_f$ and $g(x) = m(L_x^* G)$ for $x \in \mathcal{G}$. Then*

$$m(G^* F) = \int f(x) \bar{g}(x) dx.$$

Proof. Since $G^* \in L_1(\Gamma)$, there exist H and K in $L_2(\Gamma)$ such that $G^* = HK^*$. Let $H = L_h$ and $K = L_k$ where h and k are in $L_2(\mathcal{G})$. Then for any $T \in \mathfrak{L}$,

$$\begin{aligned} m(G^*T) &= m(HK^*T) = m(THK^*) = (L_{Th}, L_k) && \text{by Lemma 9.5} \\ &= (Th, k) && \text{by Corollary 9.3.} \end{aligned}$$

Now

$$m(G^*F) = (Fh, k) = \int f(x)(L_x h, k) dx = \int f(x) m(G^* L_x) dx = \int f(x) \bar{g}(x) dx.$$

COROLLARY 9.7. *Let* $F \in L_1(\Gamma) \cap L_2(\Gamma)$ *and set* $f(x) = m(L_x^* F)$ *for* $x \in \mathcal{G}$. *Then* $f \in L_2(\mathcal{G})$ *and* $F = L_f$.

Proof. Since $F \in L_2(\Gamma)$, $F = L_g$ for some $g \in L_2(\mathcal{G})$. If $h \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$, then

$$\int g(x)\bar{h}(x)dx = m(L_h^* L_g) \qquad \text{by Corollary 8.3}$$

$$= \int f(x)\bar{h}(x)dx \qquad \text{by Theorem 8.6.}$$

In particular $\int_S g(x)dx = \int_S f(x)dx$ for every measurable subset S of $\mathcal{G}$ with finite measure. It follows that $f = g$ a.e.

The representation $L \otimes L$ of $\mathcal{G}$ is a multiple of the left regular representation L; in fact, $L \otimes L$ is an $\aleph$-fold copy of L where $\aleph$ is the dimension of $\mathcal{H}$. This means that the representation $a \rightarrow L_a \otimes I$ where I is the identity on $\mathcal{H}$ is unitarily equivalent to the representation $L \otimes L$. A particular unitary operator which implements this equivalence is the operator W on $L_2(\mathcal{G} \times \mathcal{G}) = \mathcal{H} \otimes \mathcal{H}$ defined by $(Wf)(x, y) = f(x, xy)$. If $T \in \mathfrak{L}$, let $\Phi(T) = W^{-1}(T \otimes I)W$. Then Φ is a $*$-isomorphism of $\mathfrak{L}$ onto the W^*-algebra generated by $L \otimes L$, such that $\Phi(L_a) = L_a \otimes L_a$ for $a \in \mathcal{G}$. Furthermore Φ is weakly continuous on the unit ball of $\mathfrak{L}$, since Φ^{-1} is obviously weakly continuous and the unit balls of $\mathfrak{L}$ and $\Phi(\mathfrak{L})$ are compact in the weak topology.

In the case of an abelian group $\mathcal{G}$, for $t \in L_1(\mathcal{G})$, the operator L_t corresponds to multiplication by the Fourier transform $\hat{t}$ of t on $L_2(\widehat{\mathcal{G}})$ where $\widehat{\mathcal{G}}$ is the character group of $\mathcal{G}$. An easy computation shows that $\Phi(L_t)$ corresponds to multiplication by the function $\hat{t}(\hat{x}\hat{y})$ of two variables $\hat{x}$ and $\hat{y}$ on $L_2(\widehat{\mathcal{G}} \times \widehat{\mathcal{G}})$. If F and G are functions in $L_1(\widehat{\mathcal{G}})$, their convolution is the function $F*G$ satisfying the equation

$$\int (F * G)(\hat{x})\hat{t}(\hat{x})d\hat{x} = \iint F(\hat{x})G(\hat{y})\hat{t}(\hat{x}\hat{y})d\hat{x}d\hat{y}.$$

Thus, when $\mathcal{G}$ is an arbitrary unimodular group, we are led to the following definition of convolution in $L_1(\Gamma)$.

DEFINITION. If F and G are in $L_1(\Gamma)$, we define $F*G$ to be the unique element of $L_1(\Gamma)$ such that

$$m((F * G)T) = (m \times m)((F \otimes G)\Phi(T)) \qquad \text{for all } T \in \mathfrak{L}$$

where $m \times m$ is the product gage on $\mathfrak{L} \otimes \mathfrak{L}$. That such an operator $F*G$ exists is a consequence of [10, Theorem 14] and the fact that the right-hand side is weakly continuous (and hence strongly continuous) in T when T is restricted to the unit ball of $\mathfrak{L}$.

This convolution in $L_1(\Gamma)$ back transforms into pointwise multiplication over $\mathcal{G}$, as it should.

LEMMA 9.8. *Let F and G be in $L_1(\Gamma)$. Set $f(x)=m(L_x^*F)$ and $g(x)=m(L_x^*G)$ for $x\in\mathcal{G}$. Then $f(x)g(x)=m(L_x^*(F*G))$.*

Proof.

$$\begin{aligned} f(x)g(x) &= m(L_x^*F)m(L_x^*G) = (m\times m)((L_x^*F)\otimes(L_x^*G)) \\ &= (m\times m)((L_x^*\otimes L_x^*)(F\otimes G)) \\ &= (m\times m)(\Phi(L_x^*)(F\otimes G)) = m(L_x^*(F*G)). \end{aligned}$$

COROLLARY 9.9. *The set of all back transforms of operators in $L_1(\Gamma)$ is an algebra under pointwise multiplication.*

Along with the convolution in the gage space just defined, there is a companion involution. Let C denote complex conjugation in $\mathcal{H}=L_2(\mathcal{G})$; i.e., $(Cf)(x)=\bar{f}(x)$. If T is any operator on $\mathcal{H}$, we define $\tilde{T}$ to be CTC. It is easy to see that $\tilde{L}_a=L_a$ for $a\in\mathcal{G}$, and therefore $T\to\tilde{T}$ is a conjugate-linear $*$-automorphism of $\mathcal{L}$. If we set $m'(P)=m(\tilde{P})$ when P is a projection in $\mathcal{L}$, then m' is a gage on $\mathcal{L}$. Furthermore, $L_1(\Gamma)=L_1(\mathcal{H},\ \mathcal{L},\ m')$ and $m'(T)=[m(\tilde{T})]^-$ when T is integrable in either sense. When $f\in L_1(\mathcal{G})$, $\tilde{L}_f=L_{\bar{f}}$, and so $m(L_f)=[m(\tilde{L}_f)]^-$. Therefore, $m=m'$ by Corollary 1.3. We conclude that $T\to\tilde{T}$ gives involutions of $L_1(\Gamma)$ and $L_2(\Gamma)$.

LEMMA 9.10. *Let $F\in L_1(\Gamma)$. Set $f(x)=m(L_x^*F)$ for $x\in\mathcal{G}$. Then $\bar{f}(x)=m(L_x^*\tilde{F})$*

Proof. $\bar{f}(x)=[m(L_x^*F)]^-=m([L_x^*F]^\sim)=m(\tilde{L}_x^*\tilde{F})=m(L_x^*\tilde{F})$.

COROLLARY 9.11. *The set of all back transforms of operators in $L_1(\Gamma)$ is closed under complex conjugation.*

LEMMA 9.12. *If $F\in L_1(\Gamma)$ and $G\in L_1(\Gamma)\cap L_2(\Gamma)$, then $F*G\in L_2(\Gamma)$ and $\|F*G\|_2\leqq\|F\|_1\|G\|_2$.*

Proof. Let $H=L_h$ where $h\in L_1(\mathcal{G})\cap L_2(\mathcal{G})$. Then

$$m(H(F*G)) = (m\times m)(\Phi(H)(F\otimes G)) = m(\xi_2(\Phi(H)(F\otimes G)))$$

in the notation of §7. But by Theorem 8.4,

$$\xi_2(\Phi(H)(F\otimes G)) = RF$$

where $(Rf,f)=m({}^f\Phi(H)G)$. Now

$$\begin{aligned} ({}^f\Phi(H)g,g) &= ((H\otimes I)W(f\otimes g),\ W(f\otimes g)) \\ &= \int\int \bar{f}(x)\bar{g}(xy)dxdy\int h(z)f(z^{-1}x)g(z^{-1}xy)dz \\ &= \int \bar{g}(y)dy\int h(z)\left[\int \bar{f}(x)f(z^{-1}x)dx\right]g(z^{-1}x)dz \\ &= (L_{h(f^*f^*)^-}g,\ g). \end{aligned}$$

Therefore ${}^{f}\Phi(H)=L_{h(f*f^*)^-}$. That means that

$$
\begin{aligned}
|(Rf, g)| &= |m(L_{h(g*f^*)^-}G)| \leqq \|G\|_2 \|h(g*f^*)^-\|_2 \\
&\leqq \|G\|_2 \|h\|_2 \|g\|_2 \|f\|_2.
\end{aligned}
$$

Hence

$$\|R\|_\infty \leqq \|G\|_2 \|h\|_2 = \|G\|_2 \|H\|_2.$$

Now we have

$$|m(H(F*G))| = |m(RF)| \leqq \|F\|_1 \|G\|_2 \|H\|_2.$$

This inequality holds for all H of the form $H=L_h$ where $h\in L_1(\mathcal{G})\cap L_2(\mathcal{G})$. It follows from Remark 9.4 together with Remark 9.14 below that $F*G \in L_2(\Gamma)$ and $\|F*G\|_2 \leqq \|F\|_1\|G\|_2$.

Corollary 9.13. *The set of all back transforms of operators in $L_1(\Gamma)\cap L_2(\Gamma)$ is a $*$-algebra under pointwise multiplication and complex conjugation.*

Remark 9.14. In this remark we interrupt the notation of this section. Suppose that $(\mathcal{H}, \mathcal{A}, m)$ is any nonsingular gage space, and that $\mathfrak{M}$ is a $*$-subalgebra which is contained in $L_2(\mathcal{H}, \mathcal{A}, m)$ and is dense in $L_2(\mathcal{H}, \mathcal{A}, m)$. Then $\mathfrak{M}$ is strongly dense in $\mathcal{A}$. The proof is copied from the relevant portion in the proof of Theorem 1 in [11]. Let $\overline{\mathfrak{M}}$ be the strong closure of $\mathfrak{M}$. The Hilbert space $L_2(\mathcal{H}, \overline{\mathfrak{M}}, m)$ consists of those elements of $L_2(\mathcal{H}, \mathcal{A}, m)$ which are affiliated with $\overline{\mathfrak{M}}$. But by [10, Theorem 13], $L_2(\mathcal{H}, \overline{\mathfrak{M}}, m)$ is a *closed* subspace of $L_2(\mathcal{H}, \mathcal{A}, m)$, and hence $L_2(\mathcal{H}, \overline{\mathfrak{M}}, m)=L_2(\mathcal{H}, \mathcal{A}, m)$ since $\mathfrak{M}$ is dense in $L_2(\mathcal{H}, \mathcal{A}, m)$. It follows that $\overline{\mathfrak{M}}=\mathcal{A}$.

Resuming the notation of this section, we now prove analogues of the Fourier inversion theorem.

Theorem 9.15. *If $f\in L_1(\mathcal{G})$ and $F=L_f\in L_1(\Gamma)$, then $f(x)=m(L_x^*F)$ a.e.*

Proof. Since F is both bounded and in $L_1(\Gamma)$, $F\in L_2(\Gamma)$. Let $F=L_h$ where $h\in L_2(\mathcal{G})$. Let G be any element of $L_1(\Gamma)\cap L_2(\Gamma)$ and set $g(x)=m(L_x^*G)$, $x\in\mathcal{G}$. Then

$$\int f(x)\bar{g}(x)dx = m(G*F) \qquad \text{by Theorem 9.6}$$

$$= \int h(x)\bar{g}(x)dx \qquad \text{by Corollary 9.3 and Corollary 9.7.}$$

Hence $f=h$ a.e. by Corollary 9.13 and Remarks 9.4 and 9.14. Therefore $f(x) =m(L_x^*F)$ a.e. by Corollary 9.7.

Corollary 9.16. *If f is a continuous positive definite function in $L_1(\Gamma)$, then $F=L_f\in L_1(\Gamma)$ and $f(x)=m(L_x^*F)$ for all $x\in\mathcal{G}$.*

Proof. Since $F \in L_1(\Gamma)$ by Theorem 9.2, Theorem 9.15 gives $f(x) = m(L_x^* F)$ a.e. But both sides of this equation are continuous in x.

THEOREM 9.17. *Let $F \in L_1(\Gamma)$. Set $f(x) = m(L_x^* F)$ for $x \in \mathcal{G}$. If $f \in L_1(\mathcal{G})$, then $F = L_f$.*

Proof. Since f is both bounded and in $L_1(\mathcal{G})$, $f \in L_2(\mathcal{G})$. Let g be any function in $L_1(\mathcal{G}) \cap L_2(\mathcal{G})$ and set $G = L_g$. Then

$$\begin{aligned} m(G^*F) &= \int f(x)\bar{g}(x)dx && \text{by Theorem 9.6} \\ &= m(G^*L_f) && \text{by Corollary 9.3.} \end{aligned}$$

Hence $F = L_f$ by Remarks 9.4 and 9.14.

We sum up these results by writing $f \sim\sim F$ if a function f on $\mathcal{G}$ is related to a measurable operator F in any one of the three senses we have considered; i.e., if any one of the following holds:

(1) $f \in L_1(\mathcal{G})$ and $F = L_f$,
(2) $f \in L_2(\mathcal{G})$ and $F = L_f$,
(3) $F \in L_1(\Gamma)$ and $f(x) = m(L_x^* F)$ a.e.

THEOREM 9.18. *The relation $\sim\sim$ is one-to-one (if we identify functions on $\mathcal{G}$ which are equal a.e.).*

Proof. PART I. $f \sim\sim F$ and $f \sim\sim G$.

CASE (1-1). $f \in L_1(\mathcal{G})$, $F = L_f$, $G = L_f$. Then $F = G$.

CASE (1-2). $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$, $F = L_f$, $G = L_f$. Then $F = G$.

CASE (1-3). $f \in L_1(\mathcal{G})$, $F = L_f$, $G \in L_1(\Gamma)$, $f(x) = m(L_x^* G)$ a.e. Then $G = L_f = F$ by Theorem 9.17.

CASE (2-2). $f \in L_2(\mathcal{G})$, $F = L_f$, $G = L_f$. Then $F = G$.

CASE (2-3). $f \in L_2(\mathcal{G})$, $F = L_f$, $G = L_1(\Gamma)$, $f(x) = m(L_x^* G)$ a.e. Let $h \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$ and set $H = L_h$. Then

$$\begin{aligned} m(H^*F) &= \int f(x)\bar{h}(x)dx && \text{by Corollary 9.3} \\ &= m(H^*G) && \text{by Theorem 9.6.} \end{aligned}$$

Hence $F = G$ by Remarks 9.4 and 9.14.

CASE (3-3). $F \in L_1(\Gamma)$, $G \in L_1(\Gamma)$, $f(x) = m(L_x^* F) = m(L_x^* G)$ a.e. Let $h \in L_1(\mathcal{G})$ and set $H = L_h$. Then

$$m(H^*F) = \int f(x)\bar{h}(x)dx = m(H^*G) \qquad \text{by Theorem 9.6.}$$

Hence $F = G$ since the set of all such H is a strongly dense $*$-subalgebra of $\mathcal{L}$.

PART II. $f \sim\sim F$ and $g \sim\sim F$.

CASE (1-1). $f \in L_1(\mathcal{G})$, $g \in L_1(\mathcal{G})$, $F = L_f = L_g$. Then $f = g$ a.e. by Lemma 9.1.

Case (1-2). $f \in L_1(\mathcal{G})$, $g \in L_2(\mathcal{G})$, $F = L_f = L_g$. Let $H \in L_1(\Gamma) \cap L_2(\Gamma)$ and set $h(x) = m(L_x^* H)$. Then

$$\int f(x)\bar{h}(x)dx = m(H^*F) \qquad \text{by Theorem 9.6}$$

$$= \int g(x)\bar{h}(x)dx \qquad \text{by Corollary 9.7.}$$

Hence $f = g$ a.e. by Corollary 9.13 and Remarks 9.4 and 9.14.

Case (1-3). $f \in L_1(\mathcal{G})$, $F = L_f \in L_1(\Gamma)$, $g(x) = m(L_x^* F)$ a.e. Then $f = g$ a.e. by Theorem 9.15.

Case (2-2). $f \in L_2(\mathcal{G})$, $g \in L_2(\mathcal{G})$, $F = L_f = L_g$. Then $f = g$ a.e. by Corollary 9.3.

Case (2-3). $f \in L_2(\mathcal{G})$, $F = L_f \in L_1(\Gamma)$, $g(x) = m(L_x^* F)$ a.e. Then $F \in L_2(\Gamma)$. Hence $g \in L_2(\mathcal{G})$ and $F = L_g$ by Corollary 9.7. This reduces Case (2-3) to Case (2-2).

Case (3-3). $F \in L_1(\Gamma)$, $f(x) = g(x) = m(L_x^* F)$ a.e. So $f = g$ a.e.

The Riemann-Lebesgue lemma asserts that if $\mathcal{G}$ is abelian and $f \in L_1(\mathcal{G})$, then its Fourier transform vanishes at ∞ on the character group. In the present context where $\mathcal{G}$ is an arbitrary unimodular group, there is no topological sense to saying that L_f vanishes at ∞. Nonetheless, we can assert that L_f vanishes at ∞ in the measure-theoretic sense.

Theorem 9.19. *Let $f \in L_1(\mathcal{G})$ and let $F = L_f$. Then, given any $\epsilon > 0$, there exists a projection $P \in \mathfrak{L}$, such that $m(P) < +\infty$ and $\|F(I-P)\|_\infty < \epsilon$. In fact the same holds for all operators in the C^*-algebra generated by $\{L_g : g \in L_1(\mathcal{G})\}$.*

Proof. Suppose first that $f \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$. Let $|F| = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of $|F|$. Set $P = I - E_{\epsilon/2}$. Then $m(P) < +\infty$ since $F \in L_2(\Gamma)$ and

$$\|F(I-P)\|_\infty = \| \, |F| \, (I-P)\|_\infty < \epsilon.$$

Now let F be any operator in the C^*-algebra generated by $\{L_g : g \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})\}$. Let $\|F - L_g\| < \epsilon/2$ where $g \in L_1(\mathcal{G}) \cap L_2(\mathcal{G})$. Choose a projection P such that $\|L_g(I-P)\|_\infty < \epsilon/2$ and $m(P) < +\infty$. Then

$$\|F(I-P)\|_\infty \leqq \|(F - L_g)(I-P)\|_\infty + \|L_g(I-P)\|_\infty < \epsilon.$$

10. **Duality.** In [12], it is shown that for a compact group $\mathcal{G}$, the elements of $\mathcal{G}$ are in one-to-one correspondence with the nonzero $*$-homomorphisms of the representative ring into the complex numbers. In this section we prove a similar result for arbitrary unimodular groups. The algebra we consider is the set of back transforms of $L_1(\Gamma)$, which is larger than the representative ring when $\mathcal{G}$ is compact. We continue the notation of §9.

Theorem 10.1. *Consider $L_1(\Gamma)$ as a commutative Banach algebra with an involution, where convolution is the product, and the involution is $F \rightarrow \tilde{F}$. There* *is a one-to-one correspondence between operators S in $\mathfrak{L}$ which satisfy the equations*

$$S \otimes S = \Phi(S) \quad \text{and} \quad S = \tilde{S},$$

and $*$-homomorphisms σ of $L_1(\Gamma)$ into the complex numbers; the correspondence being given by $\sigma(F) = m(S^*F)$ for all F in $L_1(\Gamma)$.

Proof. Suppose that σ is a $*$-homomorphism of $L_1(\Gamma)$ into the complex numbers. By [7, §26F] σ is automatically a *continuous* linear functional on $L_1(\Gamma)$. Hence by [10, Corollary 18.1], $\sigma(F) = m(S^*F)$ for some S in $\mathfrak{L}$. Then for all F and G in $L_1(\Gamma)$,

$$\begin{aligned}(m \times m)((F \otimes G)\Phi(S^*)) &= m((F * G)S^*) = \sigma(F * G)\\ &= \sigma(F)\sigma(G) = m(FS^*)m(GS^*)\\ &= (m \times m)((F \otimes G)(S^* \otimes S^*))\end{aligned}$$

by Theorem 8.4.

Therefore, $\Phi(S) = S \otimes S$. Also for all F in $L_1(\Gamma)$,

$$m(\tilde{S}F) = [m(S\tilde{F})]^- = m(S^*\tilde{F}^*) = \sigma(\tilde{F}^*) = [\sigma(F^*)]^- = [m(S^*F^*)]^- = m(SF).$$

Hence $S = \tilde{S}$.

Conversely, suppose $S \otimes S = \Phi(S)$ and $S = \tilde{S}$. Set $\sigma(F) = m(S^*F)$. Then

$$\begin{aligned}\sigma(F * G) &= m(S^*(F * G)) = (m \times m)(\Phi(S^*)(F \otimes G))\\ &= (m \times m)((S^* \otimes S^*)(F \otimes G)) = m(S^*F)m(S^*G)\\ &= \sigma(F)\sigma(G)\end{aligned}$$

and

$$\sigma(\tilde{F}) = m(S^*\tilde{F}) = [m(S^*F)]^- = [\sigma(F)]^-.$$

THEOREM 10.2. *Any nonzero operator $S \in \mathfrak{L}$ which satisfies $S \otimes S = \Phi(S)$ is unitary.*

Proof. CASE I. $S \geqq 0$. We show first that S cannot have 0 as an approximate proper value. For suppose there exists $f_n \in \mathfrak{H}$ such that $\|f_n\| = 1$ and $\|Sf_n\| \to 0$. Take $g_0 \in \mathfrak{H}$ so that $g_1 = Sg_0 \neq 0$. Then since $S \otimes S = \Phi(S) = W^{-1}(S \otimes I)W$, we have for all f and g in $\mathfrak{H}$

$$\begin{aligned}(W(S \otimes S)(f \otimes f_n), g_0 \otimes g) &= ((S \otimes I)W(f \otimes f_n), g_0 \otimes g)\\ &= (W(f \otimes f_n), g_1 \otimes g)\\ &= \iint f(x)f_n(xy)\bar{g}_1(x)\bar{g}(y)dxdy.\end{aligned}$$

But $|(W(S \otimes S)(f \otimes f_n), g_0 \otimes g)| \leqq \|S\|_\infty \|g_0\| \, \|Sf_n\| \, \|f\| \, \|g\|$. It follows that

$$\iint |f_n(xy)\bar{g}_1(x)|^2 dxdy = \iint |f_n(y)\bar{g}_1(x)|^2 dxdy$$
$$= \|f_n\|^2\|g_1\|^2 \leqq [\|S\|_\infty \|g_0\| \|Sf_n\|]^2 \to 0$$

which is contrary to the assumption $\|f_n\| = 1$.

It follows that S is invertible. Now $\|S\| = \|\Phi(S)\| = \|S \otimes S\| = \|S\|^2$ and therefore $\|S\| = 1$. But $S^{-1} \in \mathfrak{L}$ and S^{-1} satisfies the same equation as S. Hence also $\|S^{-1}\| = 1$. Hence $S = I$.

Case II. *General.* Both S^*S and SS^* satisfy the hypotheses for Case I. Therefore $S^*S = SS^* = I$; that is, S is unitary.

Definition. We denote by Γ' the set of all nonzero operators $S \in \mathfrak{L}$ such that $S \otimes S = \Phi(S)$ and $S = \tilde{S}$.

Corollary 10.3. *The set Γ' is a group which is locally compact in the weak operator topology. The mapping $x \to L_x$ is a homeomorphism of $\mathcal{G}$ into Γ'.*

Proof. By its very definition the set $\Gamma' \cup \{0\}$ is closed under multiplication and adjunction. It follows from Theorem 10.2, that Γ' is a group. It is clear that $\Gamma' \cup \{0\}$ is a weakly closed subset of the unit ball of $\mathfrak{L}$, and therefore weakly compact. Hence Γ' itself is locally compact. The mapping $x \to L_x$ is continuous from $\mathcal{G}$ into Γ'. To show that it is a homeomorphism we use the fact that the weak operator topology on Γ' is the same as the strong operator topology. Let N be any neighborhood of e in $\mathcal{G}$. Choose $f \in \mathcal{K}$ so that $\|f\|_2 = 1$ and the support K of f is such that $KK^{-1} \subset N$. Suppose $\|L_a f - f\|_2 < 1$ for some $a \in \mathcal{G}$. Then a cannot lie outside $KK^{-1} \subset N$; for otherwise $L_a f \perp f$ and hence $\|L_a f - f\|_2 = 2^{1/2}$. Thus $x \to L_x$ is a homeomorphism.

Theorem 10.4. *The mapping $x \to L_x$ is an isomorphism of $\mathcal{G}$ onto Γ'.*

Proof. By Corollary 10.3, the image $\mathcal{G}'$ of $\mathcal{G}$ under the map $x \to L_x$ is a locally compact subgroup of Γ'. It follows that $\mathcal{G}'$ is a closed subgroup of Γ' since it is complete in the left uniform structure. Let r be the measure on Γ' which is concentrated on $\mathcal{G}'$ and is the Haar measure of $\mathcal{G}'$ there; that is to say, if f is a continuous function with compact support on Γ', then

$$\int_{\Gamma'} f(S)dr(S) = \int_{\mathcal{G}} f(L_x)dx.$$

Let $\mathcal{K}$ be the algebra of all functions g on Γ' of the form $g(S) = m(S^*G)$ for some $G \in L_1(\Gamma) \cap L_2(\Gamma)$. Consider the algebra $\mathcal{K}^2$ which consists of all sums of products of functions in $\mathcal{K}$. If $f \in \mathcal{K}^2$, then

$$f = \sum_{j=1}^{n} g_j \bar{h}_j \text{ where } g_j(S) = m(S^*G_j) \text{ and } h_j(S) = m(S^*H_j)$$

with G_j and H_j in $L_1(\Gamma) \cap L_2(\Gamma)$. Then by Corollary 9.7, $f \in L_1(\Gamma', r)$ and

$$\int f(S)dr(S) = \sum_{j=1}^{n} \int g_j(L_x)[h_j(L_x)]^- dx = \sum_{j=1}^{n} m(H_j^* G_j).$$

Furthermore, for $T \in \Gamma'$, $g_j(T^{-1}S) = m(S^*TG_j)$ and $h_j(T^{-1}S) = m(S^*TH_j)$. It follows that the left translate of f through T is also in $\mathcal{K}^2$ and

$$\begin{aligned} \int f(T^{-1}S)dr(S) &= \sum_{j=1}^{n} m((TH_j)^*(TG_j)) \\ &= \sum_{j=1}^{n} m(H_j^* G_j) = \int f(S)dr(S). \end{aligned}$$

By applying Corollary 1.4 to r and its left translates, we conclude that r is left Haar measure on Γ'. In other words $\mathcal{G}' = \Gamma'$.

COROLLARY 10.5. *Let $F \in L_1(\Gamma)$ and set $f(x) = m(L_x^* F)$ for $x \in \mathcal{G}$. Then f vanishes at ∞ on $\mathcal{G}$.*

Proof. Since $\Gamma' \cup \{0\}$ is the 1-point compactification of Γ', the function $f_1(S) = m(S^*F)$ vanishes at ∞ on Γ'.

COROLLARY 10.6. *Let ρ be a $*$-representation of $L_1(\Gamma)$ on a Hilbert space. Then there exists a $*$-representation ρ_1 of $L_\infty(\mathcal{G})$ such that $\rho(F) = \rho_1(f)$ where $f(x) = m(L_x^* F)$. In other words there exists a spectral measure E on $\mathcal{G}$ such that $\rho(F) = \int m(L_x^* F)dE(x)$.*

This follows from the discussion in [7, §26F].

DEFINITION. If μ is any complex finite regular measure on $\mathcal{G}$, we write $L_\mu = \int L_x d\mu(x)$, where the integral is taken in the weak sense. An operator $A \in \mathfrak{L}$ will be called *positive definite* if for all $F \in L_1(\Gamma)$, $m((F * \tilde{F})T) \geqq 0$.

LEMMA 10.7. *Let μ be a complex finite regular measure on $\mathcal{G}$ and let $F \in L_1(\Gamma)$. Set $f(x) = m(L_x^* F)$ for $x \in \mathcal{G}$. Then*

$$\int \bar{f}(x)d\mu(x) = m(F^* L_\mu).$$

Proof. See the proof of Theorem 9.6.

THEOREM 10.8. *The mapping $\mu \to L_\mu$ gives a one-to-one correspondence between non-negative finite regular measures on $\mathcal{G}$ and positive definite operators in $\mathfrak{L}$.*

Proof. (*Adapted from* [2]). Lemma 10.7 shows that L_μ is positive definite when μ is a non-negative finite regular measure on $\mathcal{G}$.

We consider the mapping $\mu \to L_\mu$ for all complex finite regular measures μ on $\mathcal{G}$. First we show that this mapping is one-to-one. For suppose $L_\mu = 0$. Then by Lemma 10.7, $\int \bar{f}(x)d\mu(x) = 0$ for all f which are back transforms of operators in $L_1(\Gamma)$.

But the set of all such f is an algebra to which the Stone-Weierstrass theorem applies. Hence $\mu=0$.

We equip the measures with the weak topology induced by the continuous functions vanishing at ∞ on $\mathcal{G}$, and we equip $\mathcal{L}$ with the weak topology induced by $L_1(\Gamma)$. Then Lemma 10.7 shows that $\mu \rightarrow L_\mu$ is continuous. Let C be the set of non-negative measures μ on $\mathcal{G}$ with $\mu(\mathcal{G}) \leqq 1$, and let $C' = \{L_\mu : \mu \in C\}$. If μ is a point mass of 1 at a, then $L_\mu = L_a$. Hence C' contains $\Gamma' \cup \{0\}$. Thus by the Krein-Milman theorem, it is sufficient to show that $\Gamma' \cup \{0\}$ is the set of extreme points of the set C_1 of all positive definite operators with norm $\leqq 1$. But C_1 is mapped into the set C_1' of all non-negative linear functionals of norm $\leqq 1$ on the Banach algebra $L_1(\Gamma)$ under a linear transformation which sends $\Gamma' \cup \{0\}$ into the $*$-homomorphisms of $L_1(\Gamma)$ into the complex numbers. Hence it is sufficient to show that the extreme points of C_1' are the $*$-homomorphisms of $L_1(\Gamma)$ into the complex numbers. But this follows easily from the arguments in [2].

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